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Second Quantization Approach to Quantum Hall Skyrmions

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CONTENTS

PREFACE

Skyrmions representing classical solutions to non-linear sigma model were originally introduced in nuclear physics to be identified with nucleons. Though the relevance of skyrmions to nuclear physics remains obscure, their existence is firmly established in quantum Hall physics. Namely, charged excitations observed in quantum Hall systems carry nontrivial spin textures described by skyrmions. Besides, quantum Hall skyrmions are endowed with such an utterly mathematical and fancy feature as the non-commutative geometry. This circumstance turns quantum Hall systems into the only cases where the ideas of non-commutative geometry come true in a real physical phenomena.

Quantum Hall effect carries an appealing opportunity of interplay between condensed matter physics and particle physics. Many constructions which are associated mostly with field-theoretic formalism find successful applications in quantum Hall systems. Therefore, alongside with particle physics, the quantum Hall physics represents alternative and fertile soil for demonstrating and teaching various tools of quantum field theory. In particular, one encounters the method of second quantization, basics of non-commutative geometry, W-algebras, Grassmannian fields with their non-commutative counterparts included, topological considerations, spontaneous symmetry breaking and other constructions widely employed in modern theoretical and mathematical physics. These topics are all introduced and exploited in the given lectures within the framework of quantum Hall skyrmions.

Problems supplied in the end of each chapter will help to develop and enhance the skills of practical calculations. Recommended references:

- Z.F. Ezawa, "*Quantum Hall Effects: Field Theoretical Approach and Related Topics*" (WS, Singapore 2008).
- M.O. Goerbig, "*Quantum Hall Effects*" (lecture notes), arXiv:0909.1998v2 (2009).
- J.A. Harvey, "*Komaba Lectures on Non-commutative Solitons and D-Branes*", arXiv:hep-th/0102076v1 (2001).

Derivations of some formulae in the text employ the following table of integrals and series

- A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, "*Integrals and Series*" (in Russian), vol.2 (FizMatLit 2003).

Chapter 1

INTRODUCTION

Quantum Hall (QH) effect is one of those rare physical phenomena where the quantum laws of nature can be observed on macroscopic scale. Below we first comment on the classical picture of the Hall effect and then pass to its quantum counterpart discovered by von Klitzing *et al.* in 1980.

1.1 Classical Hall Effect

The Hall effect implies that planar electrons subject to in-plane electric field $\boldsymbol{E} = (E_x, E_y, 0)$ and orthogonal magnetic field $\boldsymbol{B} = (0, 0, B_\perp)$ flow along the direction orthogonal to $\boldsymbol{E}$. In that case the net Lorentz force vanishes $\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B}) = 0$, and electrons travel with constant drift velocity $\boldsymbol{v} = |\boldsymbol{B}|^{-2}(\boldsymbol{E} \times \boldsymbol{B})$. Assuming the planar density of electrons is n_e, the linear density of the in-plane current is found to be

$$\boldsymbol{\mathcal{J}} = (en_e/\boldsymbol{B}^2)(\boldsymbol{E} \times \boldsymbol{B}) \qquad \Rightarrow \qquad \mathcal{J}_m = (en_e/B_\perp)\epsilon_{mn}E_n. \tag{1.1}$$

On the other hand the density of current and the electric field are interrelated as

$$\mathcal{J}_x = \sigma_{xx}E_x + \sigma_{xy}E_y, \tag{1.2a}$$

$$\mathcal{J}_y = \sigma_{yx}E_x + \sigma_{yy}E_y, \tag{1.2b}$$

where σ_{mn} is the conductivity tensor. The inverse of these relations is given by

$$E_x = \varrho_{xx}\mathcal{J}_x + \varrho_{xy}\mathcal{J}_y, \tag{1.3a}$$

$$E_y = \varrho_{yx}\mathcal{J}_x + \varrho_{yy}\mathcal{J}_y, \tag{1.3b}$$

where $\varrho_{mn} = (\sigma^{-1})_{mn}$ is called the resistivity tensor.

Comparing (1.2) and (1.3) to (1.1) we find the conductivity and resistivity tensors for the Hall effect to be

$$\sigma = \begin{pmatrix} 0 & \sigma_H \\ -\sigma_H & 0 \end{pmatrix} \qquad \Rightarrow \qquad \varrho = \begin{pmatrix} 0 & -1/\sigma_H \\ 1/\sigma_H & 0 \end{pmatrix} \tag{1.4}$$

where σ_H is given by

$$\sigma_H = \frac{en_e}{B_\perp} \tag{1.5}$$

and is referred to as the Hall conductivity. Remark, that σ_H depends linearly on the ratio $n_e/B_\perp$.

1.2 Quantum Hall Effect

The device where the quantum Hall effect was detected is shown schematically on figure 1.1. The system is subject to orthogonal magnetic field $\boldsymbol{B} = (0, 0, B_\perp)$. Injecting the current $\boldsymbol{J} = (0, J_y, 0)$ in y-direction, the longitudinal (V_L) and Hall (V_H) voltages are measured. These are related to the in-plane electric fields as $V_H = E_x d_x$ and $V_L = E_y d_y$ where $d_{x,y}$ are the linear sizes of a sample. Then the relations (1.3) can be rewritten as $V_H = \varrho_{xy}J_y$ and $V_L = (d_y/d_x)\varrho_{yy}J_y$, where from ϱ_{xy} and ϱ_{yy}, hence σ_{xy} and σ_{yy} can be determined.

Measurements at low temperatures ($\sim 4K$) carried out by von Klitzing *et al.* indicated that in contrast to classical picture the Hall conductivity as a function of $n_e/B_\perp$ exhibits plateaux (flat regions) as shown in figure 1.1. The corresponding values in units of $(2\pi\hbar)^{-1}e^2$ were measured to be integers with the accuracy of about 10^{-8} thus indicating the quantization of Hall conductivity. This phenomenon was named as the integer quantum Hall effect.

Later on the QH plateaux were found at fractional values of $(2\pi\hbar/e^2)\sigma_H$ with odd denominators, and such cases are referred to as the fractional quantum Hall effects. The later are much more complicated phenomena compared to integer quantum Hall effect, and the corresponding second quantization approach is still the pending challenge.

Must be pointed out that the first indications of these phenomena had been appeared in earlier theoretical studies by Ando, Matsumoto and Uemura (1975), and also in the experiments by Kawaji and Wakabayashi (1976), however none of those groups had apparently foreseen this was the quantization effect.

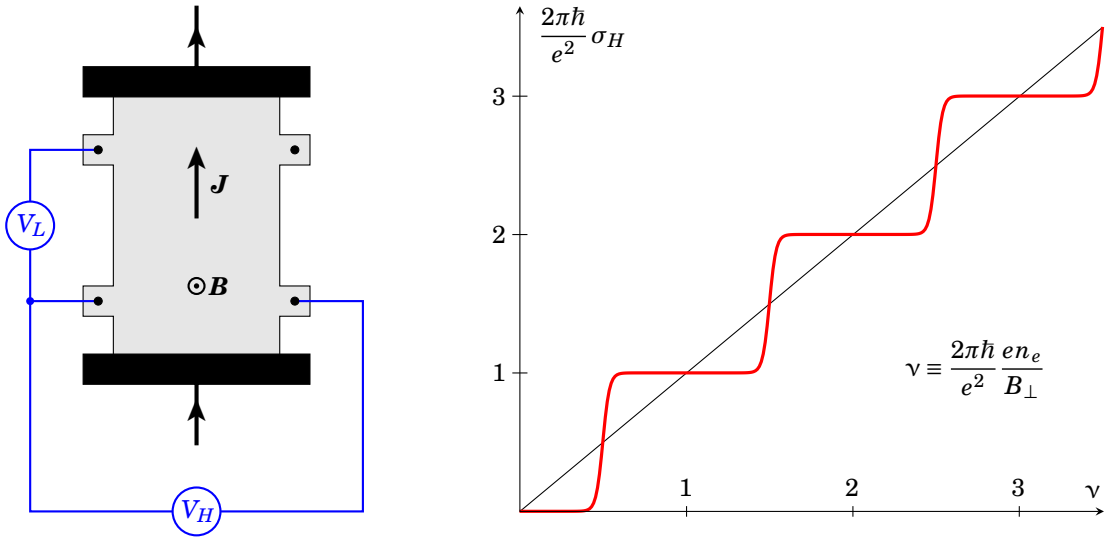


Fig. 1.1: **Left panel**: quantum Hall device. Electric current $\boldsymbol{J}$ is injected in y-direction and the voltages V_H and V_L are measured. **Right panel**: Hall conductivity σ_H versus $n_e/B_\perp$. Black curve corresponds to classical theory, while the red curve depicts the Hall conductivity at low temperatures as found by von Klitzing *et al.* Flat regions represent the quantum Hall states where the Hall conductivity $\sigma_H \equiv J_y/V_H$ is quantized in units of $(2\pi\hbar)^{-1}e^2$, while the longitudinal one vanishes $\sigma_{yy} = 0$. Here the variable $\nu \equiv (2\pi\hbar/e^2)(en_e/B_\perp)$ is employed, since it is more natural in quantum considerations.

Resistance and Resistivity in 2D

Resistance of a $3D$ conductor can be determined by $R = \frac{V}{J}$ after measuring the voltage V and current J. Then the *resistivity* ϱ is determined by $\varrho = \frac{W}{L}R$ where L is the distance between the points the voltage was measured, and W is the cross section orthogonal to the direction of electric current. Obviously, the coefficient $\frac{W}{L}$ takes the dimensionality of $length$. In d dimensions the quantity L is still the distance, but the dimensionality of cross section is $(length)^{d-1}$ so the coefficient $\frac{W}{L}$ takes the one of $(length)^{d-2}$. In this light the $2D$ conductor is a special case, since the resistance and resistivity are of same dimensionality. Nevertheless, the length L and the width (cross section) W may alter the relation between R and ϱ. Situation becomes interesting in the case of the Hall effect: the length L along which the voltage is measured coincides with the cross section because the current flows in direction orthogonal to the electric field. Therefore, we have $\frac{W}{L} = 1$, *i.e.* the resistance and resistivity coincide $\varrho = R$. This makes the Hall resistance (conductance) insensitive to sample sizes and results the extremely high precision (10^{-8}) of the Hall quantization phenomenon. The corresponding quantity is nowadays accepted as the resistance standard

$$R_K \equiv \frac{2\pi\hbar}{e^2} = 25812.8074555\Omega \tag{1.6}$$

and is referred to as the Klitzing constant.

Impurities and Plateaux

In a pure sample under homogeneous orthogonal magnetic field electrons are organized in Landau levels. Each Landau level is highly degenerated and the density of states is given by

$$D(E) = \frac{eB}{2\pi\hbar}\sum_{n=0}^{\infty}\delta(E - E_n) \tag{1.7}$$

where E_n is the energy of n'th Landau level, and $eB > 0$ is assumed.

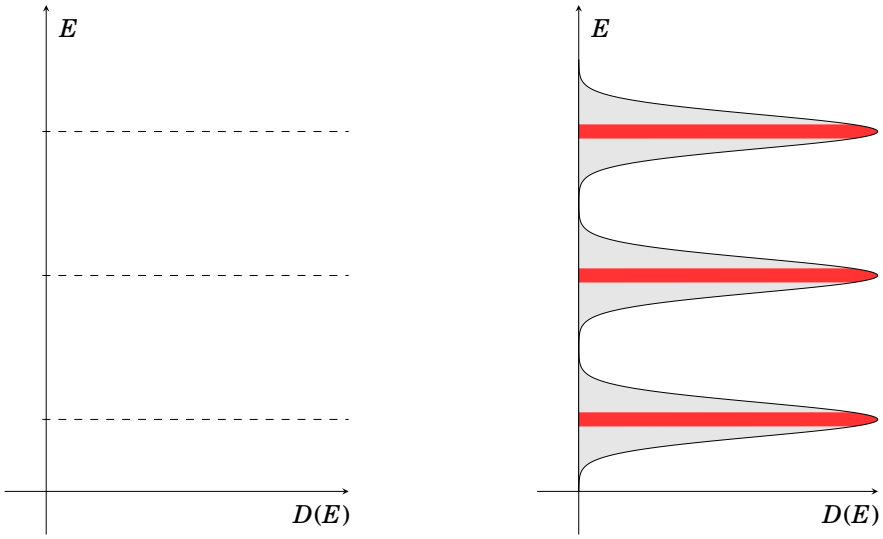


Fig. 1.2: Density of states in pure sample (left) and in the one with impurities (right).

The corresponding eigenstates are all extended states allowing electrons to freely travel across the entire sample. Therefore each electron contributes to electric current, *i.e.* increasing the electron density, the Hall conductivity must also increase. However this is not the case, and we observe wide plateaux (flat regions) around the integer values of ν as depicted in figure 1.1. Such a behaviour can be explained by the presence of impurities in real samples. Impurities modify the eigenstate content of idealized pure samples, and the density of states are no longer of δ-type, but become broadened, as shown in the right panel of figure 1.2. The density of states is now localized around $E = E_n$, with its central (red) regions representing extended states and tails (gray regions) corresponding to localized states. The later are the states where electrons are trapped by impurities, hence can reach only finite areas of a sample and therefore producing no contribution to a macroscopic electric current.

Assume now we start doping the empty system ($n_e = 0$) by electrons. Initially electrons will occupy the low-lying energy states which are localized (lowest gray), hence the initial vanishing conductance will acquire no gain until the localized sub-band is completely filled. Further increase of n_e causes electrons start occupying extended states (red sub-band), the conductance becomes nonvanishing, and keeps increasing until the sub-band is filled out. This results in jump of conductance by an amount of $(2\pi\hbar)^{-1}e^2$. Adding more electrons, we find no change in conductance, until the next gray sub-band (localized states) is filled, thus producing plateau, and so on.

Edge States

Edge states represent yet another remarkable feature of quantum Hall physics. We comment on semi-classical picture of this issue and then bring its quantum-mechanical explanation. The first one is based on skipping orbits: performing rotational movement, electrons near edges bounce back after colliding with boundaries, so the net effect is the drift motion, hence the macroscopic current along the edges. The same time, electrons in the bulk rotate along closed orbits thus giving no such contribution.

Quantum-mechanical treatment can be developed by considering the system in restricted geometry, say with $-\frac{1}{2}W \leqslant x \leqslant +\frac{1}{2}W$ and $-\infty < y < +\infty$. In that case the system enjoys translational symmetry along y direction, so the wave functions can be labeled by the corresponding momentum k. Due to the restricted geometry, the Schrödinger equation must be supplied by certain boundary conditions which determine the shape of dispersion $E(k)$. Imposing Dirichlet boundary conditions $\psi(\pm\frac{1}{2}W, y) = 0$ the resulted energy dispersion takes the form shown in the right panel of figure 1.3. Energy levels comprise flat segments and become bent at certain values of k. Energy gaps between the flat segments are precisely the cyclotron energy $\hbar\omega$. Increasing the width W of a system the flat segments become wider and in the limit of $W = \infty$ reproduce the standard Landau levels. One-particle states with the values of k from the flat segment are localized in the bulk, while those from one bent region are localized at $x = -\frac{1}{2}W$ and those from the other are localized at $x = +\frac{1}{2}W$. Remark now, that the group velocity, hence the electric current, carried by one-particle state with momentum k is proportional to $E'(k)$. Consequently the bulk states (flat segment) carry no electric current due to $E'(k) = 0$, while the edge states (bent segments) comprise finite currents due to $E'(k) \neq 0$. Remark also that the currents at opposite edges flow in opposite directions, as already pointed out within semi-classical picture.

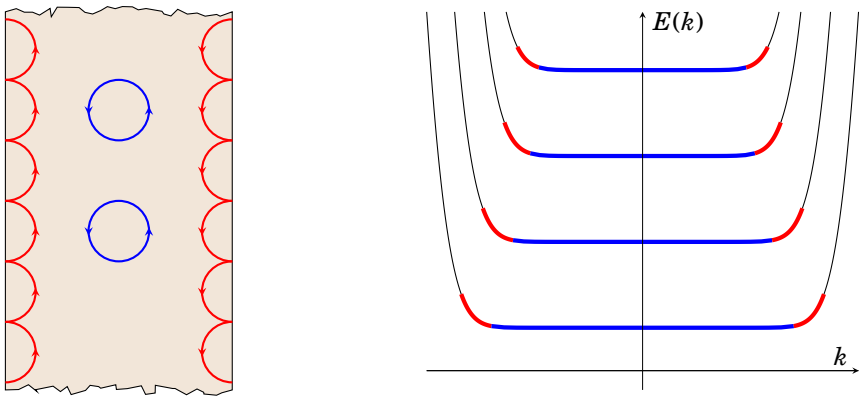


Fig. 1.3: **Left:** Semi-classical (skipping orbit) picture of edge currents. Bulk electrons rotate along closed orbits (blue) and do not contribute to macroscopic current. Electrons near edges bounce back after colliding with boundaries (red) and produce macroscopic currents localized along edges. **Right:** Energy dispersion $E(k)$ of electrons in homogeneous magnetic field and confined geometry (finite x-size).

Multicomponent Quantum Hall Systems

Most of the interesting effects take place in multicomponent QH systems where electrons carry internal degrees of freedom. The simplest example is the monolayer QH system where the internal degrees of freedom correspond to electron spin. Provided the system is subject to strong external magnetic field, electrons are organized in Landau levels with large level-splitting energy gap. For electrons in free space with the g-factor of 2 the Zeeman splitting is precisely the Landau level splitting. Would this be the case in quantum Hall samples, the energy for flipping spins would be large. Therefore, the spin degree of freedom would be frozen, hence electrons would be effectively spinless. In the solid state environment of 2-dimensional electron gas, two factors make the effective g-factor much smaller in many superconductors, particularly in GaAs samples, in which almost all quantum Hall studies have been done. The first one is the small effective mass which increases the cyclotron energy by a factor of approximately 15. The other factor is the spin-orbit coupling causing spins to tumble and reducing their coupling to external magnetic field by roughly a factor of 4. Therefore the ratio of Zeeman to cyclotron splitting is reduced from unity to about 0.02 in GaAs. Consequently the spin fluctuations become important dynamical degree of freedom leading to variety of interesting effects.

One may think that the orientations of electron spins would be random in the absence of Zeeman term ($g = 0$), however this is not the case. Exchange Coulomb interaction results in spin coherence so that in the ground state all spins become spontaneously polarized along an arbitrary direction. This implies the spontaneous breakdown of the $SU(2)$ spin symmetry, producing a gapless (Goldstone) mode. These are the low energy excitations referred to as spin waves. The system is called the quantum Hall ferromagnet. When the Zeeman coupling is turned on, the up-spin state becomes realized. The spin waves are no longer gapless, though are still the low energy excitations.

Quasiparticles (charged excitations) are topological solitons. They are vortices in spin frozen theory, as advocated by Laughlin, but are skyrmions in QH ferromagnets. Quantum Hall skyrmion is a unity of topologically nontrivial spin texture $\boldsymbol{S}(\boldsymbol{r})$ and of spatially modulated particle density $\rho(\boldsymbol{r})$. Quantum Hall skyrmions differ in several respects from standard skyrmions known in nuclear physics. The main difference is that QH skyrmions are of noncommutative nature and as an imprint of this fact they necessarily carry the electric charge.

Skyrmion excitations have been confirmed in numerous experiments by measuring the amount of reversed spins. Application of parallel magnetic field $\boldsymbol{B}_\parallel$ is an indispensable method for examining the role of the spin degrees of freedom, since it separates Zeeman and Coulomb interactions. Zeeman energy depends on the total magnetic field $B = (B_\perp^2 + B_\parallel^2)^{1/2}$, while the Coulomb energy depends only on the component $B_\perp$ orthogonal to the sample. The number of reversed spins is estimated by the increase of the Zeeman energy with the Coulomb energy being fixed.

Importance of skyrmions in connection with QH physics was theoretically recognized first in the effective Chern-Simons theory, then studied in the Hartree-Fock approximation and also by numeric analysis. In the given lectures we present the microscopic (second quantization) approach to quantum Hall skyrmions.

Chapter 2

PLANAR ELECTRON IN A HOMOGENEOUS MAGNETIC FIELD

Provided the system under consideration is subject to homogeneous magnetic field, the Landau quantization is an inevitable ingredient of our account. For the sake of completeness we first recapitulate the details of classical picture.

2.1 Classical Mechanics

Classical motion of an electron in orthogonal homogeneous magnetic field $\boldsymbol{B} = (0,0,B)$ is governed by the equations

$$m\dot{v}_x = +eBv_y, \tag{2.1a}$$

$$m\dot{v}_y = -eBv_x, \tag{2.1b}$$

which after integration give

$$v_x = C\sin(\omega t - \phi), \tag{2.2a}$$

$$v_y = C\cos(\omega t - \phi), \tag{2.2b}$$

where $\omega = eB/m$ is the cyclotron frequency. Integrating once again we come to

$$x = X_x \underbrace{-(C/\omega)\cos(\omega t - \phi)}_{R_x}, \tag{2.3a}$$

$$y = X_y \underbrace{+(C/\omega)\sin(\omega t - \phi)}_{R_y}, \tag{2.3b}$$

where $\boldsymbol{X} = (X_x, X_y)$ is referred to as the *guiding center coordinate* specifying the location of the circle along which the electron circulates. Coordinate of the electron relative to the center is denoted by $\boldsymbol{R} = (R_x, R_y)$ with $|\boldsymbol{R}| = |C/\omega|$. Obviously, the absolute coordinate is given by $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{X}$. Mind that the energy of an electron depends only on $|\boldsymbol{R}|$ but not on $\boldsymbol{X}$.

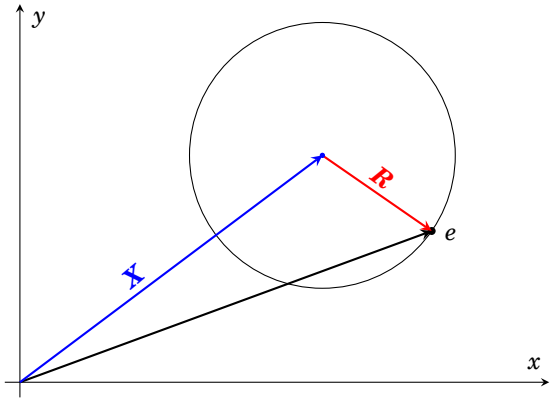


Fig. 2.1: Classical electron performing circular motion around the guiding centre.

2.2 Quantum Mechanics

We comment on quantum-mechanical picture of an electron in magnetic field. Quantum-mechanical kinetic Hamiltonian is given by

$$H = \frac{1}{2m_e}(\boldsymbol{p} - e\boldsymbol{A})^2, \tag{2.4}$$

where $\boldsymbol{p} = -i\hbar(\partial_x, \partial_y)$ is the momentum operator, and $\boldsymbol{A} = (A_x, A_y)$ describes the homogeneous orthogonal magnetic field $B = \partial_x A_y - \partial_y A_x$.

For our purposes it is more convenient to work in symmetric gauge $\boldsymbol{A} = \frac{1}{2}B(-y, x)$. Then the Hamiltonian (2.4) takes the form

$$H = \frac{1}{2m_e}(-i\hbar\partial_x + \tfrac{1}{2}eBy)^2 + \frac{1}{2m_e}(-i\hbar\partial_y - \tfrac{1}{2}eBx)^2. \tag{2.5}$$

We assume $eB > 0$ and introduce the so called magnetic length ℓ as follows

$$\frac{1}{\ell^2} = \frac{eB}{\hbar}. \tag{2.6}$$

We then rewrite (2.5) as

$$H = \frac{\hbar^2}{2m_e\ell^4}(\tfrac{1}{2}x + i\ell^2\partial_y)^2 + \frac{\hbar^2}{2m_e\ell^4}(\tfrac{1}{2}y - i\ell^2\partial_x)^2. \tag{2.7}$$

Remind that in classical picture the energy depends only on relative coordinates $\boldsymbol{R}$ but not on $\boldsymbol{X}$. In this light we identify the two operators located in square braces in (2.7) with relative coordinate operators

$$\begin{aligned} R_x &= \tfrac{1}{2}x + i\ell^2\partial_y, \\ R_y &= \tfrac{1}{2}y - i\ell^2\partial_x. \end{aligned} \tag{2.8}$$

The classical relation $\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{X}$ must be held also for the corresponding quantum-mechanical operators. Consequently, we guess the guiding centre operators as

$$\begin{aligned} X_x &= \tfrac{1}{2}x - i\ell^2\partial_y, \\ X_y &= \tfrac{1}{2}y + i\ell^2\partial_x. \end{aligned} \tag{2.9}$$

Now the kinetic Hamiltonian is rewritten as

$$H = \frac{\hbar^2}{2m_e\ell^4}\boldsymbol{R}^2. \tag{2.10}$$

Commutation relations among $\boldsymbol{R}$ and $\boldsymbol{X}$ are given by

$$[R_x, R_y] = +i\ell^2, \tag{2.11a}$$

$$[X_x, X_y] = -i\ell^2, \tag{2.11b}$$

$$[R_i, X_j] = 0. \tag{2.11c}$$

indicating that the pairs (R_x, R_y) and (X_x, X_y) form the two independent oscillator algebras.

Introduce the ladder operators

$$a = \frac{R_x + iR_y}{\sqrt{2}\ell}, \qquad a^\dagger = \frac{R_x - iR_y}{\sqrt{2}\ell}, \qquad \Rightarrow \qquad [a, a^\dagger] = 1, \tag{2.12a}$$

$$b = \frac{X_x - iX_y}{\sqrt{2}\ell}, \qquad b^\dagger = \frac{X_x + iX_y}{\sqrt{2}\ell}, \qquad \Rightarrow \qquad [b, b^\dagger] = 1. \tag{2.12b}$$

Then the kinetic Hamiltonian appears as

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}), \qquad \omega = \frac{\hbar}{m_e\ell^2}. \tag{2.13}$$

Eigenstates are classified by two integers associated with two independent oscillator algebras and look as

$$|m,n\rangle = \|m\rangle\!\rangle \otimes |n\rangle = \left[\frac{(a^\dagger)^m}{\sqrt{m!}}\,\|0\rangle\!\rangle\right] \otimes \left[\frac{(b^\dagger)^n}{\sqrt{n!}}\,|0\rangle\right], \tag{2.14}$$

where $\|0\rangle\!\rangle$ and $|0\rangle$ are defined by $a\|0\rangle\!\rangle = 0$ and $b|0\rangle = 0$ respectively.

Eigenvalues are given by

$$H|m,n\rangle = \hbar\omega(m+\tfrac{1}{2})|m,n\rangle, \tag{2.15}$$

and electrons are organized in the equidistant energy levels called the Landau levels.

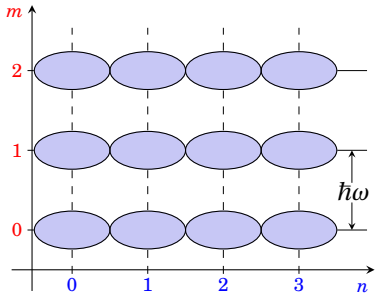


Fig. 2.2: Electrons in homogeneous magnetic field are organized in Landau levels. Each circle denotes the quantum states labelled by a pair (m,n) where m specifies the energy level. Each Landau level is degenerated and the quantum states within a given level are distinguished by the values of n.

Provided the Hamiltonian (2.13) commutes with b and $b^\dagger$, each Landau level is highly degenerated. The power of degeneracy is characterized by the maximal affordable density of identical electrons within each level. Later on we show this density is given by $\rho_L \equiv (2\pi\ell^2)^{-1}$.

Magnetic fields characteristic for quantum Hall regime are strong enough so that the Landau gap $\hbar\omega$ becomes large compared to other energy scales related with Coulomb and Zeeman interactions. Then the excitations across the Landau levels are suppressed and electrons become confined to the lowest Landau level (LLL). Degrees of freedom associated with the algebra (2.11a) become frozen, and the kinematic Hamiltonian is quenched. Consequently, kinematics of the system becomes governed solely by the algebra (2.11b), signifying that the electron coordinates do not commute among each other. This is the manifestation of non-commutativity at the level of quantum mechanics.

For the wave function of an electron in the LLL we have

$$a|0,n\rangle = 0 \qquad \Longrightarrow \qquad \left[\tfrac{1}{2}(x+iy) + \ell^2(\partial_x + i\partial_y)\right]\psi_{LLL}(\boldsymbol{r}) = 0. \tag{2.16}$$

Introducing $\sqrt{2}z = (x+iy)/\ell$ and $\sqrt{2}\bar{z} = (x-iy)/\ell$ we rewrite (2.16) as

$$\left[\frac{z}{2} + \frac{\partial}{\partial\bar{z}}\right]\psi_{LLL}(\bar{z},z) = 0. \tag{2.17}$$

Searching for $\psi_{LLL}(\bar{z},z)$ in the form

$$\psi_{LLL}(\bar{z},z) = \exp(-\tfrac{1}{2}\bar{z}z)f(\bar{z},z) \tag{2.18}$$

we come to

$$\frac{\partial}{\partial\bar{z}}f(\bar{z},z) = 0, \tag{2.19}$$

i.e. $f(\bar{z},z)$ must be the function of z only.

Therefore the LLL eigenstate $|0,n\rangle$ in $\boldsymbol{r}$-representation can be written as

$$\langle\boldsymbol{r}|0,n\rangle = \exp(-\tfrac{1}{2}\bar{z}z)f_n(z), \tag{2.20}$$

and due to $b|0,0\rangle = 0$ we must have

$$\left[\frac{\bar{z}}{2} + \frac{\partial}{\partial z}\right]\exp(-\tfrac{1}{2}\bar{z}z)f_0(z) = 0 \qquad \Longrightarrow \qquad \frac{\partial}{\partial z}f_0(z) = 0. \tag{2.21}$$

Consequently (including the normalization factor)

$$\langle \boldsymbol{r}|0,0\rangle = \frac{1}{\sqrt{2\pi\ell^2}}\exp(-\tfrac{1}{2}\bar{z}z). \tag{2.22}$$

Acting on this function repeatedly by the operator $b^\dagger$ we obtain

$$\langle \boldsymbol{r}|0,n\rangle = \frac{1}{\sqrt{n!}}\langle \boldsymbol{r}|(b^\dagger)^n|0,0\rangle = \frac{1}{\sqrt{n!}}\left[\frac{z}{2}-\frac{\partial}{\partial\bar{z}}\right]^n \frac{1}{\sqrt{2\pi\ell^2}}\exp(-\tfrac{1}{2}\bar{z}z) =$$

$$= \frac{z^n}{\sqrt{2\pi\ell^2 n!}}\exp(-\tfrac{1}{2}\bar{z}z) = \frac{1}{\sqrt{2\pi\ell^2 n!}}\left[\frac{x+iy}{\sqrt{2}\,\ell}\right]^n e^{-\frac{1}{4}\ell^{-2}(x^2+y^2)}. \tag{2.23}$$

Mind that the wave functions $\langle \boldsymbol{r}|0,n\rangle$ are orthogonal but do not form the complete set because the states $\langle \boldsymbol{r}|m,n\rangle$ with $m>0$ are absent. For this reason electrons in the lowest Landau level cannot be localized to a point but are smeared over some minimal area in the plane. This area is related to the order of degeneracy of Landau levels.

In order to calculate these characteristics we consider the probability distribution for the wave function in the lowest Landau level. It appears as

$$|\langle \boldsymbol{r}|0,n\rangle|^2 = \frac{1}{2\pi\ell^2 n!}\left[\frac{r^2}{2\ell^2}\right]^n e^{-\frac{1}{2}\ell^{-2}r^2} \tag{2.24}$$

and is localized along the circle of radius $r_n=\sqrt{2n}\,\ell$. The ones for $n=0$ and $n=1$ are shown in figure 2.3.

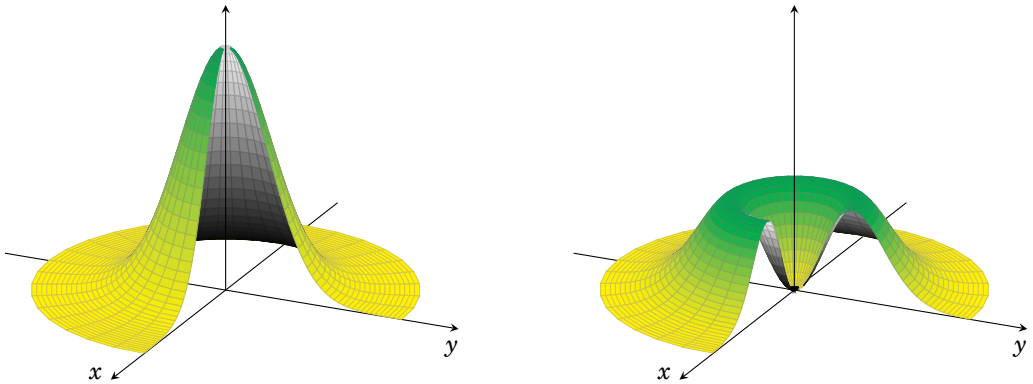


Fig. 2.3: $|\langle \boldsymbol{r}|0,0\rangle|^2$ (left), $|\langle \boldsymbol{r}|0,1\rangle|^2$ (right).

In order to calculate the order of degeneracy consider a disc of the radius between r_N and r_{N+1}. Then the peaks of $|\langle \boldsymbol{r}|0,n\rangle|^2$ with $n=0,1,2,\ldots,N$ fall in the interior of the disc while the ones of $|\langle \boldsymbol{r}|0,n\rangle|^2$ with $n>N$ are beyond. Thus the $N+1$ states occupy the area $\mathcal{A}$ with $2\pi\ell^2 N<\mathcal{A}<2\pi\ell^2(N+1)$. The area per quantum state is given by $\mathcal{A}/N$ and in the limit of infinitely large disc ($N\to\infty$) takes the value $\mathcal{A}/N\to 2\pi\ell^2$. Then the quantity

$$\rho_L \equiv \frac{1}{2\pi\ell^2} \tag{2.25}$$

measures the order of degeneracy of LLL by specifying the maximal density of electrons that can be accommodated in there.

Problems

(2.a) Verify the commutation relations (2.11) by use of (2.8) and (2.9).

(2.b) Verify the wave functions (2.23) are orthonormal.

Chapter 3

BASICS OF NON-COMMUTATIVE GEOMETRY

Noncommutativity occurs in planar electron systems where the physics develops in the lowest Landau level (LLL). Confinement of electrons to the LLL results in the non-commutativity of electron position operators.

Original idea of non-commutative coordinates dates back to Heisenberg, who employed this construction for avoiding the electron self-energy singularities. Subsequent developments led to elaboration of the non-commutative multiplication law introduced first by Gröenewold (1946) and then by Moyal (1949).

The essence of non-commutative geometry becomes clearer when formulated in algebraic terms. Commutativity of a plane implies the algebra of smooth functions over the plane with the algebraic operation to be the ordinary product. The plane with the coordinates $\boldsymbol{r} \equiv (x, y)$ is said to be non-commutative if the algebraic operation is defined by the Groenewold-Moyal product

$$f_1(\boldsymbol{r}) \star f_2(\boldsymbol{r}) = f_1(\boldsymbol{r})\, e^{-(i/2)\theta\epsilon_{nj}\overleftarrow{\partial}_n\overrightarrow{\partial}_j} f_2(\boldsymbol{r}), \tag{3.1}$$

where $\overrightarrow{\partial}$ stands for the derivative acting on the right, while $\overleftarrow{\partial}$ acts on left. In this sense the formal expression (3.1) is understood as the following power expansion

$$f_1 \star f_2 = f_1 f_2 + \tfrac{1}{1!}(-\tfrac{i}{2}\theta)^1\epsilon_{nj}(\partial_n f_1)(\partial_j f_2) + \tfrac{1}{2!}(-\tfrac{i}{2}\theta)^2\epsilon_{mi}\epsilon_{nj}(\partial_m\partial_n f_1)(\partial_i\partial_j f_2) + \cdots, \tag{3.2}$$

where the quantity θ is the parameter determining the "magnitude" of non-commutativity.

The non-commutative product (3.1) is *the only possible* deformation of ordinary (commutative) product satisfying the associativity

$$[f_1(\boldsymbol{r}) \star f_2(\boldsymbol{r})] \star f_3(\boldsymbol{r}) = f_1(\boldsymbol{r}) \star [f_2(\boldsymbol{r}) \star f_3(\boldsymbol{r})]. \tag{3.3}$$

Obviously, for $\theta = 0$ it is reduced back to the ordinary product.

From (3.1) we find

$$x \star y - y \star x = -i\theta, \tag{3.4}$$

indicating that the coordinates of a plane are non-commutative with respect to the algebraic multiplication law. In this scope they resemble the operators $\boldsymbol{X} = (X_x, X_y)$ which form the oscillator algebra (with $\theta = \ell^2$)

$$X_x X_y - X_y X_x = -i\theta. \tag{3.5}$$

The mapping $\boldsymbol{r} \mapsto \boldsymbol{X}$ generates the one of a function $f(\boldsymbol{r})$ into the corresponding operator $O[f]$ referred to as the Weyl operator. It is given by

$$O[f] = \frac{1}{2\pi}\int \Bigg[\underbrace{\frac{1}{2\pi}\int f(\boldsymbol{r}) e^{-i\boldsymbol{k}\boldsymbol{r}} d\boldsymbol{r}}_{f(\boldsymbol{k})}\Bigg] e^{i\boldsymbol{k}\boldsymbol{X}} d\boldsymbol{k}, \tag{3.6}$$

where the quantity in square braces is the Fourier transformation of $f(\boldsymbol{r})$. The function $f(\boldsymbol{r})$ is usually referred to as the symbol of $O[f]$.

Let $O[f_1]$ and $O[f_2]$ be the Weyl operators of $f_1(\boldsymbol{r})$ and $f_2(\boldsymbol{r})$ respectively, and consider the product

$$O[f_1]O[f_2] = \left[\frac{1}{2\pi}\int f_1(\boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{X}} d\boldsymbol{k}\right]\left[\frac{1}{2\pi}\int f_2(\boldsymbol{k}') e^{i\boldsymbol{k}'\boldsymbol{X}} d\boldsymbol{k}'\right] =$$

$$= \frac{1}{4\pi^2}\int f_1(\boldsymbol{k}) f_2(\boldsymbol{k}') e^{i\boldsymbol{k}\boldsymbol{X}} e^{i\boldsymbol{k}'\boldsymbol{X}} d\boldsymbol{k} d\boldsymbol{k}'. \tag{3.7}$$

Due to (3.5) we have $[\boldsymbol{k}\boldsymbol{X}, \boldsymbol{k}'\boldsymbol{X}] = \boldsymbol{k} \wedge \boldsymbol{k}'$ where $\boldsymbol{k} \wedge \boldsymbol{k}'$ is the c-number. Then the Baker-Campbell-Hausdorff formula yields

$$e^{i\boldsymbol{k}\boldsymbol{X}} e^{i\boldsymbol{k}'\boldsymbol{X}} = e^{i(\theta/2)\boldsymbol{k}\wedge\boldsymbol{k}'} e^{i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{X}}. \tag{3.8}$$

Using this in (3.7) and shifting the integration variable $\boldsymbol{k} \to \boldsymbol{k} - \boldsymbol{k}'$ we pass to

$$O[f_1]O[f_2] = \frac{1}{4\pi^2}\int \Big\{ \int f_1(\boldsymbol{k}-\boldsymbol{k}')f_2(\boldsymbol{k}')e^{i(\theta/2)\boldsymbol{k}\wedge\boldsymbol{k}'}d\boldsymbol{k}' \Big\} e^{i\boldsymbol{k}\boldsymbol{X}}d\boldsymbol{k}. \tag{3.9}$$

On the other hand calculating the Fourier transform of $f_1(\boldsymbol{r}) \star f_2(\boldsymbol{r})$ we find

$$\int e^{-i\boldsymbol{k}\boldsymbol{r}} \Big\{ f_1(\boldsymbol{r}) \star f_2(\boldsymbol{r}) \Big\} d\boldsymbol{r} = \int e^{-i\boldsymbol{k}\boldsymbol{r}} \left\{ \left[\frac{1}{2\pi}\int f_1(\boldsymbol{k}')e^{+i\boldsymbol{k}'\boldsymbol{r}}d\boldsymbol{k}' \right] \star \left[\frac{1}{2\pi}\int f_2(\boldsymbol{k}'')e^{+i\boldsymbol{k}''\boldsymbol{r}}d\boldsymbol{k}'' \right] \right\} d\boldsymbol{r} =$$

$$= \frac{1}{4\pi^2}\int e^{-i\boldsymbol{k}\boldsymbol{r}} f_1(\boldsymbol{k}')f_2(\boldsymbol{k}'') \Big\{ e^{+i\boldsymbol{k}'\boldsymbol{r}} \star e^{+i\boldsymbol{k}''\boldsymbol{r}} \Big\} d\boldsymbol{k}'d\boldsymbol{k}''d\boldsymbol{r}. \tag{3.10}$$

For calculating the non-commutative product in square braces we use $\partial_n e^{i\boldsymbol{k}\boldsymbol{r}} = ik_n e^{i\boldsymbol{k}\boldsymbol{r}}$ and obtain

$$e^{+i\boldsymbol{k}'\boldsymbol{r}} \star e^{+i\boldsymbol{k}''\boldsymbol{r}} = e^{+i\boldsymbol{k}'\boldsymbol{r}} e^{-(i/2)\theta\epsilon_{nj}\overleftarrow{\partial}_n\overrightarrow{\partial}_j} e^{+i\boldsymbol{k}''\boldsymbol{r}} = e^{+i\boldsymbol{k}'\boldsymbol{r}} e^{+(i/2)\theta\boldsymbol{k}'\wedge\boldsymbol{k}''} e^{+i\boldsymbol{k}''\boldsymbol{r}}. \tag{3.11}$$

Substituting (3.11) into (3.10) and integrating over $\boldsymbol{r}$ we pass to

$$\int e^{-i\boldsymbol{k}\boldsymbol{r}} \Big\{ f_1(\boldsymbol{r}) \star f_2(\boldsymbol{r}) \Big\} d\boldsymbol{r} = \frac{1}{4\pi^2}\int e^{-i\boldsymbol{k}\boldsymbol{r}} f_1(\boldsymbol{k}')f_2(\boldsymbol{k}'')e^{+i\boldsymbol{k}'\boldsymbol{r}+(i/2)\theta\boldsymbol{k}'\wedge\boldsymbol{k}''+i\boldsymbol{k}''\boldsymbol{r}} d\boldsymbol{k}'d\boldsymbol{k}''d\boldsymbol{r} =$$

$$= \int f_1(\boldsymbol{k}')f_2(\boldsymbol{k}'')e^{+(i/2)\theta\boldsymbol{k}'\wedge\boldsymbol{k}''}d\boldsymbol{k}'d\boldsymbol{k}'' \underbrace{\frac{1}{4\pi^2}\int e^{-i(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'')\boldsymbol{r}}d\boldsymbol{r}}_{\delta(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'')} =$$

$$= \int f_1(\boldsymbol{k}-\boldsymbol{k}'')f_2(\boldsymbol{k}'')e^{+(i/2)\theta\boldsymbol{k}\wedge\boldsymbol{k}''}d\boldsymbol{k}''. \tag{3.12}$$

Comparing the integrand of (3.9) to (3.12) we write

$$O[f_1]O[f_2] = \frac{1}{2\pi}\int \left[\frac{1}{2\pi}\int \Big\{ f_1(\boldsymbol{r}) \star f_2(\boldsymbol{r}) \Big\} e^{-i\boldsymbol{k}\boldsymbol{r}}d\boldsymbol{r} \right] e^{i\boldsymbol{k}\boldsymbol{X}}d\boldsymbol{k}, \tag{3.13}$$

where the right hand side is $O[f_1 \star f_2]$, as it follows from the definition (3.6). We thus obtain the important relation

$$O[f_1]O[f_2] = O[f_1 \star f_2], \tag{3.14}$$

which generalizes the interplay between (3.4) and (3.5).

Weyl operators act in the space of states where the basis states can be set up by use of ladder operators

$$b = \frac{X_x - iX_y}{\sqrt{2\theta}}, \qquad b^\dagger = \frac{X_x + iX_y}{\sqrt{2\theta}}, \qquad [b\,,b^\dagger] = 1, \tag{3.15}$$

which up to $\theta = \ell^2$ are in fact the ones already introduced by (2.12b).

Holomorphic basis is defined as

$$|n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}}|0\rangle, \qquad n = 0,1,\ldots \tag{3.16}$$

where $|0\rangle$ is the vacuum state $b|0\rangle = 0$.

Using (3.15) and (3.16) we obtain

$$\langle n|e^{+i\boldsymbol{k}\boldsymbol{X}}|n+\alpha\rangle = \frac{(i\ell)^\alpha\sqrt{n!}}{2^{\alpha/2}\sqrt{(n+\alpha)!}}(k_x+ik_y)^\alpha e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}L_n^\alpha(\tfrac{1}{2}\ell^2\boldsymbol{k}^2), \tag{3.17}$$

where L_n^α is the generalized Laguerre polynomial. Then

$$\mathrm{Tr}\big[e^{i\boldsymbol{k}\boldsymbol{X}}\big] = \sum_{n=0}^{\infty}\langle n|e^{i\boldsymbol{k}\boldsymbol{X}}|n\rangle = e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}\sum_{n=0}^{\infty}L_n(\tfrac{1}{2}\ell^2\boldsymbol{k}^2). \tag{3.18}$$

In order to estimate the right hand side we present it as

$$\sum_{n=0}^{\infty} L_n(\tfrac{1}{2}\ell^2\boldsymbol{k}^2) = \lim_{t\to 1}\sum_{n=0}^{\infty} t^n L_n(\tfrac{1}{2}\ell^2\boldsymbol{k}^2) = \lim_{t\to 1}\frac{1}{1-t}e^{-\frac{1}{2}\frac{t}{1-t}\ell^2\boldsymbol{k}^2}, \tag{3.19}$$

where $t<1$ and the last equality holds due to (5.11.2.1) of *Prudnikov vol. 2*. Using this in (3.18) we obtain

$$\mathrm{Tr}\big[e^{i\boldsymbol{k}\boldsymbol{X}}\big] = \lim_{t\to 1}\frac{1}{1-t}e^{-\frac{1}{4}\frac{1+t}{1-t}\ell^2\boldsymbol{k}^2} = \begin{cases} \infty & \boldsymbol{k}=0 \\ \\ 0 & \boldsymbol{k}\neq 0 \end{cases} \tag{3.20}$$

thus resembling the Dirac's delta. On the other hand we have

$$\int \frac{1}{1-t}e^{-\frac{1}{4}\frac{1+t}{1-t}\ell^2\boldsymbol{k}^2}d\boldsymbol{k} = \frac{4\pi\ell^{-2}}{1+t} \overset{t\to 1}{\longrightarrow} \frac{2\pi}{\ell^2}, \tag{3.21}$$

what enables to put

$$\mathrm{Tr}\big[e^{i\boldsymbol{k}\boldsymbol{X}}\big] = 2\pi\ell^{-2}\delta(\boldsymbol{k}). \tag{3.22}$$

We use this relation to determine the Fourier transform $f(\boldsymbol{k})$ from the Weyl operator $O[f]$. Write (3.6) as

$$O[f] = \frac{1}{2\pi}\int f(\boldsymbol{k}')e^{i\boldsymbol{k}'\boldsymbol{X}}d\boldsymbol{k}'. \tag{3.23}$$

Multiplying this relation by $e^{-i\boldsymbol{k}\boldsymbol{X}}$ and taking trace we find

$$\mathrm{Tr}\Big\{e^{-i\boldsymbol{k}\boldsymbol{X}}O[f]\Big\} = \frac{1}{2\pi}\int f(\boldsymbol{k}')\,\mathrm{Tr}\Big\{e^{-i\boldsymbol{k}\boldsymbol{X}}e^{i\boldsymbol{k}'\boldsymbol{X}}\Big\}d\boldsymbol{k}' =$$

$$= \frac{1}{2\pi}\int f(\boldsymbol{k}')e^{-i(\theta/2)\boldsymbol{k}\wedge\boldsymbol{k}'}\mathrm{Tr}\Big\{e^{-i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{X}}\Big\}d\boldsymbol{k}' =$$

$$= \frac{1}{2\pi}\int f(\boldsymbol{k}')e^{-i(\theta/2)\boldsymbol{k}\wedge\boldsymbol{k}'}2\pi\ell^{-2}\delta(\boldsymbol{k}-\boldsymbol{k}')d\boldsymbol{k}', \tag{3.24}$$

where from we eventually come to

$$f(\boldsymbol{k}) = \ell^2\mathrm{Tr}\Big\{e^{-i\boldsymbol{k}\boldsymbol{X}}O[f]\Big\}. \tag{3.25}$$

In the end we bring several relations to be employed later. Expanding Weyl operator in $|n\rangle$-basis we write

$$O[f] = \sum_{mn} O_{mn}[f]|m\rangle\langle n|, \tag{3.26}$$

where the matrix $O_{mn}[f]$ is given by

$$O_{mn}[f] = \langle m|O[f]|n\rangle. \tag{3.27}$$

The inversion formula (3.25) reads

$$f(\boldsymbol{k}) = \theta\sum_{mn}\langle n|e^{-i\boldsymbol{k}\boldsymbol{X}}|m\rangle O_{mn}[f]. \tag{3.28}$$

Summarizing, we have the chain of one-to-one mappings

$$f(\boldsymbol{r}) \quad\longleftrightarrow\quad O[f] \quad\longleftrightarrow\quad O_{mn}[f], \tag{3.29}$$

accompanied by the ones among the multiplications laws

$$f_1(\boldsymbol{r})\star f_2(\boldsymbol{r}) \quad\longleftrightarrow\quad O[f_1]O[f_2] \quad\longleftrightarrow\quad \sum_j O_{mj}[f_1]O_{jn}[f_2]. \tag{3.30}$$

Also, we have the correspondence

$$\partial_m f(\boldsymbol{r}) \quad\longleftrightarrow\quad -\frac{i}{\theta}\epsilon_{mn}[X_n, O[f]], \tag{3.31}$$

which determines the differentiation in the space of Weyl operators.

Problems

(3.a) Employ the Fourier representation for $\delta(\boldsymbol{r})$ and show that $\delta(\boldsymbol{r})\star\delta(\boldsymbol{r}) = \pi^{-2}\theta^{-2}$.

(3.b) Employ the ladder operators (2.15) and derive (3.17).

Chapter 4

FIELD THEORY IN THE LOWEST LANDAU LEVEL

Quantum field theory, or equivalently the formalism of second quantization is a powerful method for describing the systems comprising many particles. This tool is especially helpful in the cases of finite particle densities like the one under consideration.

4.1 Field Operators and Fock States

Introduce the electron field operator in the lowest Landau level

$$\psi_\mu(\boldsymbol{r}) = \sum_{n=0}^{\infty} \langle \boldsymbol{r}|0,n\rangle c_\mu(n), \tag{4.1}$$

where $\mu = 1,2,\ldots,N$ describes the isospin (internal) degree of freedom. For illustrative simplicity we will use $\mu = \uparrow,\downarrow$ (*i.e.* $N = 2$) from time to time. Further, $\langle \boldsymbol{r}|0,n\rangle$ is the one-particle wave function in the LLL, and $c_\mu(n)$ is the electron annihilation operator. Its hermitian conjugate $c_\mu^\dagger(n)$ is the electron creation operator, and we have the standard anticommutation relations

$$\left\{c_\mu(m), c_\nu^\dagger(n)\right\} = \delta_{\mu\nu}\delta_{mn}. \tag{4.2}$$

These operators act in the space of Fock states which is built up starting from the Fock vacuum state $|\varnothing\rangle$ (we use this symbol in order to avoid confusion with one-particle vacuum state $|0\rangle$) defined as

$$c_\mu(n)|\varnothing\rangle = 0. \tag{4.3}$$

Then the basis states of Fock space can be set up by multiple applications of creation operators $c_\mu^\dagger(n)$ to the Fock vacuum. Some simple Fock states for $N = 2$ can be pictorially presented as follows

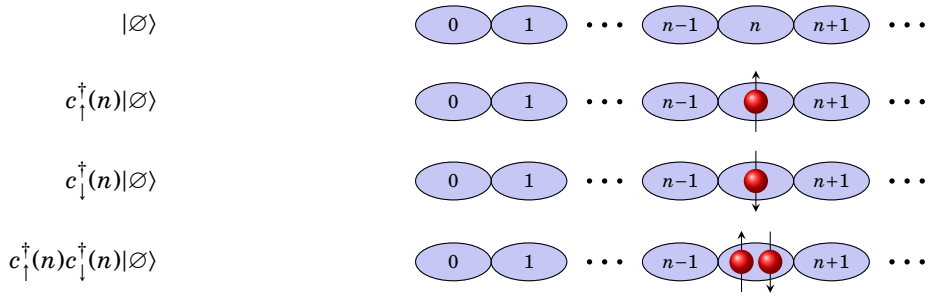
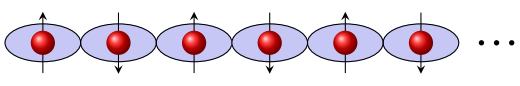


Fig. 4.1: Operator $c_\mu^\dagger(n)$ creates an electron with polarization μ at site n. Each site can accommodate up to N electrons simultaneously (here $N = 2$).

As an instructive example later on we consider the particular case of the Fock state given by

$$|\mathfrak{S}\rangle = \prod_{n=0}^{\infty} c_\uparrow^\dagger(2n)c_\downarrow^\dagger(2n+1)|\varnothing\rangle \quad \text{☛} \tag{4.4}$$

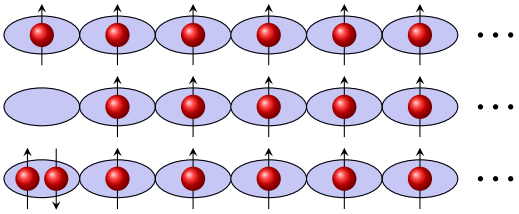


In the final part of these lectures we will discuss the physically relevant Fock states

ground state $|\mathrm{g}\rangle =$

hole-excited state $|\mathrm{h}\rangle =$

electron-excited state $|\mathrm{e}\rangle =$



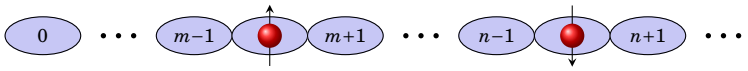
4.2 Relation to Many-Body Quantum Mechanics

We demonstrate how the Fock states are related to many-body quantum mechanical wave functions. For this purpose we employ the case of a simple Fock state

$$|\mathfrak{S}\rangle = c_{\uparrow}^{\dagger}(m)c_{\downarrow}^{\dagger}(n)|\varnothing\rangle \tag{4.5}$$

with $m < n$. Pictorially this can be presented as

$|\mathfrak{S}\rangle =$



i.e. all sites are empty except m'th and n'th.

Calculate the expectation value

$$\Psi_{\mu\nu}(\boldsymbol{r}_1,\boldsymbol{r}_2) \equiv \langle\varnothing|\psi_{\mu}(\boldsymbol{r}_1)\psi_{\nu}(\boldsymbol{r}_2)|\mathfrak{S}\rangle. \tag{4.6}$$

Substituting (4.1) we find

$$\Psi_{\mu\nu}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \langle\varnothing|\left[\sum_{s=0}^{\infty}\langle\boldsymbol{r}_1|0,s\rangle c_{\mu}(s)\right]\left[\sum_{t=0}^{\infty}\langle\boldsymbol{r}_2|0,t\rangle c_{\nu}(t)\right]c_{\uparrow}^{\dagger}(m)c_{\downarrow}^{\dagger}(n)|\varnothing\rangle =$$

$$= \sum_{st}\langle\boldsymbol{r}_1|0,s\rangle\langle\boldsymbol{r}_2|0,t\rangle\langle\varnothing|c_{\mu}(s)c_{\nu}(t)c_{\uparrow}^{\dagger}(m)c_{\downarrow}^{\dagger}(n)|\varnothing\rangle. \tag{4.7}$$

Such matrix elements are usually calculated by use of $c_{\mu}(m)|\varnothing\rangle = 0$ after dragging annihilation operators through the creation ones up to $|\varnothing\rangle$. Using anticommutation relations (4.2) for $c_{\nu}(t)c_{\uparrow}^{\dagger}(m)$ we find

$$\langle\varnothing|c_{\mu}(s)c_{\nu}(t)c_{\uparrow}^{\dagger}(m)c_{\downarrow}^{\dagger}(n)|\varnothing\rangle = \langle\varnothing|c_{\mu}(s)\Big[\delta_{\nu\uparrow}\delta_{mt} - c_{\uparrow}^{\dagger}(m)c_{\nu}(t)\Big]c_{\downarrow}^{\dagger}(n)|\varnothing\rangle =$$

$$= \delta_{\nu\uparrow}\delta_{mt}\langle\varnothing|c_{\mu}(s)c_{\downarrow}^{\dagger}(n)|\varnothing\rangle - \langle\varnothing|c_{\mu}(s)c_{\uparrow}^{\dagger}(m)c_{\nu}(t)c_{\downarrow}^{\dagger}(n)|\varnothing\rangle. \tag{4.8}$$

Performing the similar trick for $c_{\nu}(t)c_{\downarrow}^{\dagger}(n)$ in the second term we pass to

$$\langle\varnothing|c_{\mu}(s)c_{\nu}(t)c_{\uparrow}^{\dagger}(m)c_{\downarrow}^{\dagger}(n)|\varnothing\rangle = \delta_{\nu\uparrow}\delta_{mt}\langle\varnothing|c_{\mu}(s)c_{\downarrow}^{\dagger}(n)|\varnothing\rangle - \delta_{\nu\downarrow}\delta_{nt}\langle\varnothing|c_{\mu}(s)c_{\uparrow}^{\dagger}(m)|\varnothing\rangle, \tag{4.9}$$

where the two-point averages $\langle\varnothing|c_{\mu}(s)c_{\downarrow}^{\dagger}(n)|\varnothing\rangle$ and $\langle\varnothing|c_{\mu}(s)c_{\uparrow}^{\dagger}(m)|\varnothing\rangle$ can be calculated by use of the same technique and we come to

$$\langle\varnothing|c_{\mu}(s)c_{\nu}(t)c_{\uparrow}^{\dagger}(m)c_{\downarrow}^{\dagger}(n)|\varnothing\rangle = \delta_{\nu\uparrow}\delta_{mt}\delta_{\mu\downarrow}\delta_{ns} - \delta_{\nu\downarrow}\delta_{nt}\delta_{\mu\uparrow}\delta_{ms}, \tag{4.10}$$

which substituted into (4.7) leads to

$$\Psi_{\mu\nu}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \delta_{\mu\downarrow}\delta_{\nu\uparrow}\langle\boldsymbol{r}_1|0,n\rangle\langle\boldsymbol{r}_2|0,m\rangle - \delta_{\mu\uparrow}\delta_{\nu\downarrow}\langle\boldsymbol{r}_1|0,m\rangle\langle\boldsymbol{r}_2|0,n\rangle. \tag{4.11}$$

This can be written also as a Slater determinant

$$\Psi_{\mu\nu}(\boldsymbol{r}_1,\boldsymbol{r}_2) = -\begin{vmatrix} \delta_{\mu\uparrow}\langle\boldsymbol{r}_1|0,m\rangle & \delta_{\mu\downarrow}\langle\boldsymbol{r}_1|0,n\rangle \\ \delta_{\nu\uparrow}\langle\boldsymbol{r}_2|0,m\rangle & \delta_{\nu\downarrow}\langle\boldsymbol{r}_2|0,n\rangle \end{vmatrix}. \tag{4.12}$$

Thus the quantum-mechanical many-body wave functions are comprised in second quantized Fock states and can be extracted as

$$\Psi_{\mu_1\mu_2\cdots\mu_N}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_N) = \langle\varnothing|\psi_{\mu_1}(\boldsymbol{r}_1)\psi_{\mu_2}(\boldsymbol{r}_2)\cdots\psi_{\mu_N}(\boldsymbol{r}_N)|\mathfrak{S}\rangle. \tag{4.13}$$

4.3 Physical and Core Fields

Second quantized operators describing observable quantities are the particle and isospin density operators

$$\rho(\boldsymbol{r}) = \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}), \tag{4.14a}$$

$$I_a(\boldsymbol{r}) = \tfrac{1}{2}\psi^\dagger(\boldsymbol{r})\lambda_a\psi(\boldsymbol{r}), \tag{4.14b}$$

where λ_a are $N\times N$ "Gell-Mann matrices" with $a = 1,2,\ldots,N^2-1$, and N the amount of electron polarizations. For a single layer spinning electrons we have $N = 2$ with $\mu = \uparrow,\downarrow$, while for a double layer spinning electrons we have $N = 4$: up-layer up-spin, up-layer down-spin, down-layer up-spin, down-layer down-spin.

The matrices λ_a satisfy the relations

$$\big[\lambda_a,\lambda_b\big] = 2if_{abc}\lambda_c, \tag{4.15a}$$

$$\{\lambda_a,\lambda_b\} = 2d_{abc}\lambda_c + \frac{4}{N}\delta_{ab}\mathbf{1}_{N\times N}, \tag{4.15b}$$

$$\mathrm{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}, \tag{4.15c}$$

where f_{abc} are the $SU(N)$ structure constants, and $\mathbf{1}_{N\times N}$ is the $N\times N$ identity matrix.

For a given Fock state we introduce the *classical fields* representing particle and spin distributions over plane as

$$\rho_{\mathrm{cl}}(\boldsymbol{r}) = \langle\mathfrak{S}|\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})|\mathfrak{S}\rangle, \tag{4.16a}$$

$$I_a^{\mathrm{cl}}(\boldsymbol{r}) = \tfrac{1}{2}\langle\mathfrak{S}|\psi^\dagger(\boldsymbol{r})\lambda_a\psi(\boldsymbol{r})|\mathfrak{S}\rangle. \tag{4.16b}$$

Instead of dealing separately with $\rho(\boldsymbol{r})$ and $I_a(\boldsymbol{r})$, it is more convenient to use a single operator simultaneously comprising $\rho(\boldsymbol{r})$ and $I_a(\boldsymbol{r})$ in one. For this purpose we introduce

$$D_{\mu\nu}(\boldsymbol{r}) \equiv \psi^\dagger_\nu(\boldsymbol{r})\psi_\mu(\boldsymbol{r}), \tag{4.17}$$

which is related to particle and isospin operators as

$$\rho(\boldsymbol{r}) = \mathrm{Tr}\big[D(\boldsymbol{r})\big], \tag{4.18a}$$

$$I_a(\boldsymbol{r}) = \tfrac{1}{2}\mathrm{Tr}\big[\lambda_a D(\boldsymbol{r})\big], \tag{4.18b}$$

$$D_{\mu\nu}(\boldsymbol{r}) = \frac{1}{N}\delta_{\mu\nu}\rho(\boldsymbol{r}) + (\lambda_a)_{\mu\nu}I_a(\boldsymbol{r}), \tag{4.18c}$$

where the traces are implied over the isospin indices.

Obviously, identical relations are held for the corresponding classical fields, *i.e.* introducing

$$D^{\mathrm{cl}}_{\mu\nu}(\boldsymbol{r}) = \langle\mathfrak{S}|D_{\mu\nu}(\boldsymbol{r})|\mathfrak{S}\rangle, \tag{4.19}$$

we have

$$\rho^{\mathrm{cl}}(\boldsymbol{r}) = \mathrm{Tr}\big[D^{\mathrm{cl}}(\boldsymbol{r})\big], \tag{4.20a}$$

$$I_a^{\mathrm{cl}}(\boldsymbol{r}) = \tfrac{1}{2}\mathrm{Tr}\big[\lambda_a D^{\mathrm{cl}}(\boldsymbol{r})\big], \tag{4.20b}$$

$$D^{\mathrm{cl}}_{\mu\nu}(\boldsymbol{r}) = \frac{1}{N}\delta_{\mu\nu}\rho_{\mathrm{cl}}(\boldsymbol{r}) + (\lambda_a)_{\mu\nu}I_a^{\mathrm{cl}}(\boldsymbol{r}). \tag{4.20c}$$

Though the physical quantities are $D_{\mu\nu}(\boldsymbol{r})$, all the very peculiarities of the LLL field theory become manifest only in terms of *core quantities* we are now about to introduce. For this purpose we first calculate the Fourier transform

$$D_{\mu\nu}(\boldsymbol{k}) = \frac{1}{2\pi}\int D_{\mu\nu}(\boldsymbol{r})e^{-i\boldsymbol{k}\boldsymbol{r}}d\boldsymbol{r} = \frac{1}{2\pi}\int \psi^{\dagger}_{\nu}(\boldsymbol{r})\psi_{\mu}(\boldsymbol{r})e^{-i\boldsymbol{k}\boldsymbol{r}}d\boldsymbol{r}. \tag{4.21}$$

Substituting here (4.1) we find

$$\begin{aligned} D_{\mu\nu}(\boldsymbol{k}) &= \frac{1}{2\pi}\sum_{mn} c^{\dagger}_{\nu}(m)c_{\mu}(n)\int \langle 0,m|\boldsymbol{r}\rangle\langle \boldsymbol{r}|0,n\rangle e^{-i\boldsymbol{k}\boldsymbol{r}}d\boldsymbol{r} = \\ &= \frac{1}{2\pi}\sum_{mn} c^{\dagger}_{\nu}(m)c_{\mu}(n)\langle 0,m|e^{-i\boldsymbol{k}\boldsymbol{r}}|0,n\rangle = \\ &= \frac{1}{2\pi}\sum_{mn} c^{\dagger}_{\nu}(m)c_{\mu}(n)\langle 0,m|e^{-i\boldsymbol{k}(\boldsymbol{R}+\boldsymbol{X})}|0,n\rangle = \\ &= \frac{1}{2\pi}\sum_{mn} c^{\dagger}_{\nu}(m)c_{\mu}(n)\langle 0,m|e^{-i\boldsymbol{k}\boldsymbol{R}}e^{-i\boldsymbol{k}\boldsymbol{X}}|0,n\rangle. \end{aligned} \tag{4.22}$$

where the last equality holds since $\boldsymbol{R}$ commutes with $\boldsymbol{X}$.

Recalling that $|0,n\rangle = \|0\rangle\!\rangle \otimes |n\rangle$ we find

$$\langle 0,m|e^{-i\boldsymbol{k}\boldsymbol{R}}e^{-i\boldsymbol{k}\boldsymbol{X}}|0,n\rangle = \langle\!\langle 0\|e^{-i\boldsymbol{k}\boldsymbol{R}}\|0\rangle\!\rangle\langle m|e^{-i\boldsymbol{k}\boldsymbol{X}}|n\rangle. \tag{4.23}$$

Expressing $\boldsymbol{R}$ in terms of a-operators and using the Baker-Campbell-Hausdorff formula we find

$$\langle\!\langle 0\|e^{-i\boldsymbol{k}\boldsymbol{R}}\|0\rangle\!\rangle = e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}. \tag{4.24}$$

Then the Fourier transform (4.22) appears as

$$D_{\mu\nu}(\boldsymbol{k}) = e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}\hat{D}_{\mu\nu}(\boldsymbol{k}) \tag{4.25}$$

where

$$\hat{D}_{\mu\nu}(\boldsymbol{k}) \equiv \frac{1}{2\pi}\sum_{mn}\langle m|e^{-i\boldsymbol{k}\boldsymbol{X}}|n\rangle c^{\dagger}_{\nu}(m)c_{\mu}(n). \tag{4.26}$$

Because of the reasonings presented below we name this quantity as a *core quantity*. Introduce $\hat{D}_{\mu\nu}(\boldsymbol{r})$ as

$$\hat{D}_{\mu\nu}(\boldsymbol{r}) \equiv \frac{1}{2\pi}\int \hat{D}_{\mu\nu}(\boldsymbol{k})e^{+i\boldsymbol{k}\boldsymbol{r}}d\boldsymbol{k}. \tag{4.27}$$

In order to demonstrate the essence of core fields, it is instructive to derive the relation between $D_{\mu\nu}(\boldsymbol{r})$ and $\hat{D}_{\mu\nu}(\boldsymbol{r})$. For this purpose we use (4.25) and write

$$\begin{aligned} D_{\mu\nu}(\boldsymbol{r}) &= \frac{1}{2\pi}\int D_{\mu\nu}(\boldsymbol{k})e^{+i\boldsymbol{k}\boldsymbol{r}}d\boldsymbol{k} = \\ &= \frac{1}{2\pi}\int e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}\hat{D}_{\mu\nu}(\boldsymbol{k})e^{+i\boldsymbol{k}\boldsymbol{r}}d\boldsymbol{k}. \end{aligned} \tag{4.28}$$

Expressing $\hat{D}_{\mu\nu}(\boldsymbol{k})$ via $\hat{D}_{\mu\nu}(\boldsymbol{r})$ and substituting into the later we pass to

$$\begin{aligned} D_{\mu\nu}(\boldsymbol{r}) &= \frac{1}{2\pi}\int e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}\left[\frac{1}{2\pi}\int \hat{D}_{\mu\nu}(\boldsymbol{r}')e^{-i\boldsymbol{k}\boldsymbol{r}'}d\boldsymbol{r}'\right]e^{+i\boldsymbol{k}\boldsymbol{r}}d\boldsymbol{k} = \\ &= \frac{1}{4\pi^2}\int\left[\int e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}e^{+i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}d\boldsymbol{k}\right]\hat{D}_{\mu\nu}(\boldsymbol{r}')d\boldsymbol{r}'. \end{aligned} \tag{4.29}$$

Calculating the integral in square braces we obtain

$$D_{\mu\nu}(\boldsymbol{r}) = \frac{1}{\pi\ell^2}\int e^{-\ell^{-2}(\boldsymbol{r}-\boldsymbol{r}')^2}\hat{D}_{\mu\nu}(\boldsymbol{r}')d\boldsymbol{r}'. \tag{4.30}$$

Taking the average of this relation with respect to any Fock state we find

$$D^{\rm cl}_{\mu\nu}(\boldsymbol{r}) = \frac{1}{\pi\ell^2}\int e^{-\ell^{-2}(\boldsymbol{r}-\boldsymbol{r}')^2}\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r}')d\boldsymbol{r}'. \tag{4.31}$$

Integrating both sides over $\boldsymbol{r}$ we find

$$\int D^{\rm cl}_{\mu\nu}(\boldsymbol{r})d\boldsymbol{r} = \int \hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r}')d\boldsymbol{r}', \tag{4.32}$$

indicating the amount of particles and spins comprised in $D^{\rm cl}_{\mu\nu}(\boldsymbol{r})$ coincides with those comprised in $\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r})$. Therefore the relation (4.31) means that physical fields are the same core fields but smeared over the area of order of ℓ^2 by the factor of $e^{-\ell^{-2}(\boldsymbol{r}-\boldsymbol{r}')^2}$ as shown in figure 4.2. The corresponding factor in $\boldsymbol{k}$-space is the one of $e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}$ in (4.25).

Thus, whatever the Fock state $|\mathfrak{S}\rangle$ is, the quantity $\langle\mathfrak{S}|D_{\mu\nu}(\boldsymbol{r})|\mathfrak{S}\rangle$ can never be localized to a point. This is the consequence of the quantum-mechanical fact that electrons in LLL cannot be localized to a point but only to elementary area of order of ℓ^2. In contrast, the core field $\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r})$ comprises no smearing factors and therefore, can be localized to a point.[1] In this scope, the core quantities introduced via (4.27) describe kind of "cores" located inside physical electrons which are smeared. In other words, physical objects can be regarded as cores "dressed" in a LLL cloud.

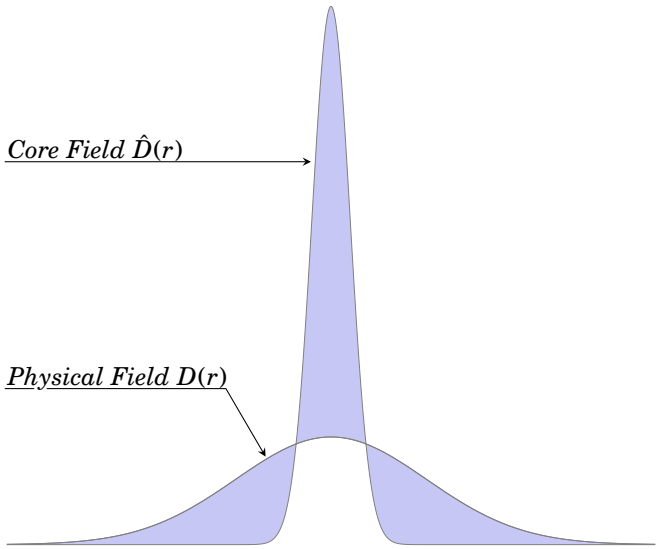


Fig. 4.2: Core fields describe the "cores" located inside the physical fields which are necessarily spatially extended.

As an explicit example of this statement we take $N = 2$ and consider the Fock state (4.4)

$$|\mathfrak{S}\rangle = \prod_{n=0}^{\infty} c^\dagger_\uparrow(2n)c^\dagger_\downarrow(2n+1)|\varnothing\rangle. \tag{4.33}$$

In order to obtain the average $D^{\rm cl}_{\mu\nu}(\boldsymbol{r}) = \langle\mathfrak{S}|D_{\mu\nu}(\boldsymbol{r})|\mathfrak{S}\rangle$ we first calculate

$$\langle\mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)|\mathfrak{S}\rangle = \begin{cases} 1 & for \quad \mu=\nu=\uparrow, \quad m=n=0,2,4,\ldots \\ 1 & for \quad \mu=\nu=\downarrow, \quad m=n=1,3,5,\ldots \\ 0 & otherwise. \end{cases} \tag{4.34}$$

[1] If more precisely, the system under consideration is the field theoretical model at *finite densities* what means that the ground state of the system differs from the vacuum state of the theory. In this light, the expectation value of $\hat{D}_{\mu\nu}(\boldsymbol{r})$ is decoupled into the ground state and the excitation parts. Localizable is the excitation part of $\langle\mathfrak{S}|\hat{D}_{\mu\nu}(\boldsymbol{r})|\mathfrak{S}\rangle$, while the excitation part of $\langle\mathfrak{S}|D_{\mu\nu}(\boldsymbol{r})|\mathfrak{S}\rangle$ is necessarily smeared.

From (4.26) we have

$$\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{k}) = \langle \mathfrak{S}|\hat{D}_{\mu\nu}(\boldsymbol{k})|\mathfrak{S}\rangle = \frac{1}{2\pi}\sum_{mn}\langle m|e^{-i\boldsymbol{k}\boldsymbol{X}}|n\rangle\langle \mathfrak{S}|c^{\dagger}_{\nu}(m)c_{\mu}(n)|\mathfrak{S}\rangle. \tag{4.35}$$

Using (4.34) and (3.17) we find the only non-vanishing components are

$$\hat{D}^{\rm cl}_{\uparrow\uparrow}(\boldsymbol{k}) = \frac{1}{2\pi}\sum_{\rm even}\langle n|e^{-i\boldsymbol{k}\boldsymbol{X}}|n\rangle = \frac{1}{2\pi}e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}\sum_{n=0}^{\infty}L_{2n}(\tfrac{1}{2}\ell^2\boldsymbol{k}^2), \tag{4.36a}$$

$$\hat{D}^{\rm cl}_{\downarrow\downarrow}(\boldsymbol{k}) = \frac{1}{2\pi}\sum_{\rm odd}\langle n|e^{-i\boldsymbol{k}\boldsymbol{X}}|n\rangle = \frac{1}{2\pi}e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}\sum_{n=0}^{\infty}L_{2n+1}(\tfrac{1}{2}\ell^2\boldsymbol{k}^2). \tag{4.36b}$$

In chapter 3 we have derived

$$\sum_{n=0}^{\infty}L_n(\tfrac{1}{2}\ell^2\boldsymbol{k}^2) = 2\pi\ell^{-2}\delta(\boldsymbol{k}). \tag{4.37}$$

Besides, (5.11.2.1) of *Prudnikov vol. 2.* yields

$$\sum_{n=0}^{\infty}(-1)^n L_n(\tfrac{1}{2}\ell^2\boldsymbol{k}^2) = \tfrac{1}{2}e^{\frac{1}{4}\ell^2\boldsymbol{k}^2}. \tag{4.38}$$

Using these in (4.36) we find

$$\hat{D}^{\rm cl}_{\uparrow\uparrow}(\boldsymbol{k}) = \frac{1}{2\ell^2}\delta(\boldsymbol{k}) + \frac{1}{8\pi} \qquad \Rightarrow \qquad \hat{D}^{\rm cl}_{\uparrow\uparrow}(\boldsymbol{r}) = \frac{1}{4\pi\ell^2} + \frac{1}{4}\delta(\boldsymbol{r}), \tag{4.39a}$$

$$\hat{D}^{\rm cl}_{\downarrow\downarrow}(\boldsymbol{k}) = \frac{1}{2\ell^2}\delta(\boldsymbol{k}) - \frac{1}{8\pi} \qquad \Rightarrow \qquad \hat{D}^{\rm cl}_{\downarrow\downarrow}(\boldsymbol{r}) = \frac{1}{4\pi\ell^2} - \frac{1}{4}\delta(\boldsymbol{r}), \tag{4.39b}$$

and further

$$\hat{\rho}^{\rm cl}(\boldsymbol{r}) = \rho_L, \tag{4.40a}$$

$$\hat{I}^{\rm cl}_x(\boldsymbol{r}) = \hat{I}^{\rm cl}_y(\boldsymbol{r}) = 0, \tag{4.40b}$$

$$\hat{I}^{\rm cl}_z(\boldsymbol{r}) = \tfrac{1}{4}\delta(\boldsymbol{r}), \tag{4.40c}$$

where $\rho_L = (2\pi\ell^2)^{-1}$ was introduced in (2.25). We see that the spin z-component is localized to a point.

Constructing the corresponding physical quantities by use of (4.30) we find

$$\rho^{\rm cl}(\boldsymbol{r}) = \rho_L, \tag{4.41a}$$

$$I^{\rm cl}_x(\boldsymbol{r}) = I^{\rm cl}_y(\boldsymbol{r}) = 0, \tag{4.41b}$$

$$I^{\rm cl}_z(\boldsymbol{r}) = \tfrac{1}{4}\rho_L e^{-\ell^{-2}\boldsymbol{r}^2}, \tag{4.41c}$$

where the spin z-component is exponentially smeared over the area of order of ℓ^2.

Finally, we point out one remarkable property of core fields. Average values of spin fields are usually subject to some normalization conditions, *e.g.* $\boldsymbol{S}^2 = 1$ in non-linear sigma models. Here, once the isospin fields are concerned, it is natural to expect similar normalization properties. However, the physical isospin fields set by (4.41) would hardly exhibit any such property due to the exponential factor. Core fields (4.40) contain $\delta(\boldsymbol{r})$-type factors which do not lead to any reasonable outcome when squared up. In contrast, if we employ the non-commutative multiplication, we find the sought for relation (here we use the result of problem (3.a))

$$\hat{I}^{\rm cl}_a \star \hat{I}^{\rm cl}_a = \frac{1}{16}\,\delta(\boldsymbol{r}) \star \delta(\boldsymbol{r}) = \frac{1}{16\pi^2\ell^4}, \tag{4.42}$$

signifying that the normalization conditions are obeyed by core fields, and these are the non-commutative normalization conditions. In the subsequent chapters we present this and related issues in details.

Problems

(4.a) Calculate $\langle\varnothing|\psi_{\mu_1}(\boldsymbol{r}_1)\psi_{\mu_2}(\boldsymbol{r}_2)\psi_{\mu_3}(\boldsymbol{r}_3)|\mathfrak{S}\rangle$ for the Fock state $|\mathfrak{S}\rangle = c^\dagger_{\nu_1}(n_1)c^\dagger_{\nu_2}(n_2)c^\dagger_{\nu_3}(n_3)|\varnothing\rangle$ and express in the form of Slater determinant.

(4.b) Calculate $\langle\!\langle 0\|e^{-i\boldsymbol{kR}}\|0\rangle\!\rangle$.

(4.c) Verify (4.34).

Chapter 5

NON-COMMUTATIVE KINEMATICS

5.1 Algebra $W_\infty(N)$

In previous chapter we introduced the core operator

$$\hat{D}_{\mu\nu}(\boldsymbol{k}) \equiv \frac{1}{2\pi}\sum_{mn}\langle n|e^{-i\boldsymbol{k}\boldsymbol{X}}|m\rangle c^\dagger_\nu(n)c_\mu(m). \tag{5.1}$$

Here we build the algebra satisfied by these quantities. We start by calculating the commutation relations for the combination of c-operators involved in (5.1). These are

$$D_{\mu\nu}(m,n) \equiv c^\dagger_\nu(n)c_\mu(m). \tag{5.2}$$

Trivial calculations by use of anticommutation relations (4.2) yield

$$\big[D_{\mu\nu}(m,n),D_{\sigma\tau}(s,t)\big] = \delta_{\mu\tau}\delta_{mt}D_{\sigma\nu}(s,n) - \delta_{\sigma\nu}\delta_{sn}D_{\mu\tau}(m,t). \tag{5.3}$$

Using this together with (3.8) we find

$$4\pi^2\big[\hat{D}_{\mu\nu}(\boldsymbol{k}),\hat{D}_{\sigma\tau}(\boldsymbol{k}')\big] = \delta_{\mu\tau}\sum_{ns}\langle n|e^{-i\boldsymbol{k}\boldsymbol{X}}e^{-i\boldsymbol{k}'\boldsymbol{X}}|s\rangle D_{\sigma\nu}(s,n) - \delta_{\sigma\nu}\sum_{mt}\langle t|e^{-i\boldsymbol{k}'\boldsymbol{X}}e^{-i\boldsymbol{k}\boldsymbol{X}}|m\rangle D_{\mu\tau}(m,t). \tag{5.4}$$

Using here

$$e^{-i\boldsymbol{k}\boldsymbol{X}}e^{-i\boldsymbol{k}'\boldsymbol{X}} = e^{+(i/2)\ell^2\boldsymbol{k}\wedge\boldsymbol{k}'}e^{-i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{X}} \tag{5.5}$$

we find

$$2\pi\big[\hat{D}_{\mu\nu}(\boldsymbol{k}),\hat{D}_{\sigma\tau}(\boldsymbol{k}')\big] = \delta_{\mu\tau}\,e^{+(i/2)\ell^2\boldsymbol{k}\wedge\boldsymbol{k}'}\hat{D}_{\sigma\nu}(\boldsymbol{k}+\boldsymbol{k}') - \delta_{\sigma\nu}\,e^{-(i/2)\ell^2\boldsymbol{k}\wedge\boldsymbol{k}'}\hat{D}_{\mu\tau}(\boldsymbol{k}+\boldsymbol{k}'). \tag{5.6}$$

Rewritten in terms of $\hat{\rho}(\boldsymbol{k})$ and $\hat{I}_a(\boldsymbol{k})$ this appears as

$$2\pi\big[\hat{\rho}(\boldsymbol{k}),\hat{\rho}(\boldsymbol{k}')\big] = 2i\,\hat{\rho}(\boldsymbol{k}+\boldsymbol{k}')\sin(\tfrac{1}{2}\ell^2\boldsymbol{k}\wedge\boldsymbol{k}'), \tag{5.7a}$$

$$2\pi\big[\hat{I}_a(\boldsymbol{k}),\hat{\rho}(\boldsymbol{k}')\big] = 2i\,\hat{I}_a(\boldsymbol{k}+\boldsymbol{k}')\sin(\tfrac{1}{2}\ell^2\boldsymbol{k}\wedge\boldsymbol{k}'), \tag{5.7b}$$

$$\begin{aligned}2\pi\big[\hat{I}_a(\boldsymbol{k}),\hat{I}_b(\boldsymbol{k}')\big] = {}& if_{abc}\hat{I}_c(\boldsymbol{k}+\boldsymbol{k}')\cos(\tfrac{1}{2}\ell^2\boldsymbol{k}\wedge\boldsymbol{k}')+\\ &+ id_{abc}\hat{I}_c(\boldsymbol{k}+\boldsymbol{k}')\sin(\tfrac{1}{2}\ell^2\boldsymbol{k}\wedge\boldsymbol{k}')+\\ &+ iN^{-1}\delta_{ab}\,\hat{\rho}(\boldsymbol{k}+\boldsymbol{k}')\sin(\tfrac{1}{2}\ell^2\boldsymbol{k}\wedge\boldsymbol{k}').\end{aligned} \tag{5.7c}$$

The algebra (5.7) is referred to as $W_\infty(N)$ since it is the $SU(N)$ extension of W_∞ which is given by (5.7a). Important physical consequence of $W_\infty(N)$ consists in the fact that particle and spin densities are intrinsically entangled. Such an entanglement disappears in commutative limit ($\ell \to 0$) and the algebra $W_\infty(N)$ becomes reduced back to the ordinary $SU(N)$. Charge-spin entanglement is pure non-commutative effect endowing charged excitations with nontrivial topology.

Taking $\boldsymbol{k}' = 0$ in (5.7a) and (5.7b) we find

$$\big[\hat{\rho}(\boldsymbol{k}),\hat{\rho}(0)\big] = \big[\hat{I}_a(\boldsymbol{k}),\hat{\rho}(0)\big] = 0, \tag{5.8}$$

i.e. $\hat{\rho}(0)$ is the Casimir operator of $W_\infty(N)$. Further, (5.1) give

$$\hat{\rho}(0) = \sum_\mu \hat{D}_{\mu\mu}(0) = \frac{1}{2\pi} \sum_\mu \sum_{mn} \langle n|m\rangle c^\dagger_\mu(n) c_\mu(m) = \frac{1}{2\pi} \sum_{\mu m} c^\dagger_\mu(m) c_\mu(m), \tag{5.9}$$

where from we read the physical meaning of $2\pi\hat{\rho}(0)$: it counts the total number of particles in a Fock state.

General elements of $W_\infty(N)$ are linear combinations of $\hat{D}_{\mu\nu}(\boldsymbol{k})$

$$W = \int w_{\mu\nu}(\boldsymbol{k}) \hat{D}_{\mu\nu}(\boldsymbol{k}) d\boldsymbol{k}, \tag{5.10}$$

where $w_{\mu\nu}(\boldsymbol{k})$ are complex-valued functions satisfying

$$[w_{\mu\nu}(\boldsymbol{k})]^* = w_{\nu\mu}(-\boldsymbol{k}), \tag{5.11}$$

as required for W to be hermitian.

The element (5.10) can be written in a different form. Using (5.1) we express it as

$$W = \sum_{\mu\nu} \xi_{\mu\nu}(m,n) c^\dagger_\nu(n) c_\mu(m), \tag{5.12}$$

where

$$\xi_{\mu\nu}(m,n) \equiv \frac{1}{2\pi} \int w_{\mu\nu}(\boldsymbol{k}) \langle n|e^{-i\boldsymbol{k}\boldsymbol{X}}|m\rangle d\boldsymbol{k}, \tag{5.13}$$

with

$$[\xi_{\mu\nu}(m,n)]^* = \xi_{\nu\mu}(n,m). \tag{5.14}$$

5.2 Non-Commutative Constraints

We comment on how the non-commutativity presented in the microscopic theory generates the constraints on classical objects. We first define the class of Fock states we work with, and then derive the non-commutative relations satisfied by the expectation values of core fields.

We consider the class of Fock states which can be written as

$$|\mathfrak{S}\rangle = e^{iW} |\mathfrak{S}_0\rangle, \tag{5.15}$$

where W is an arbitrary element of the algebra $W_\infty(N)$, and $|\mathfrak{S}_0\rangle$ is a pure state, *i.e.* of the form

$$|\mathfrak{S}_0\rangle = \Big\{ \prod_{\mu m} \big[c^\dagger_\mu(m)\big]^{\nu_\mu(m)} \Big\} |\varnothing\rangle. \tag{5.16}$$

Each $\nu_\mu(m)$ may take the values either 0 or 1 specifying whether the isospin state μ at Landau site m is empty or occupied, respectively.

The amount of electrons at Landau site m is given by

$$\nu(m) = \sum_{\mu=1}^{N} \nu_\mu(m) \tag{5.17}$$

and may take values from 0 up to N. The quantity referred to as filling factor is defined by

$$\nu = \lim_{M\to\infty} \frac{1}{M} \sum_{m=0}^{M-1} \nu(m) \tag{5.18}$$

and measures how many electrons are accommodated per site in average (common symbol for the filling factor is the Greek "nu". Here we use "upgreek" LaTeX font $\upnu$ for it in order to avoid confusion with the isospin index ν).

Though the class of states determined by (5.15) and (5.16) does not embrace the whole Fock space it is general enough and embraces all physically relevant (ground and excited) states at integer fillings ($\nu = integer$).

We now trace out the relations satisfied by classical fields $D^{\rm cl}_{\mu\nu}(\boldsymbol{r}) = \langle\mathfrak{S}|D_{\mu\nu}(\boldsymbol{r})|\mathfrak{S}\rangle$ where $|\mathfrak{S}\rangle$ is of the form (4.15). These relations will appear as generalization of the non-commutative normalization (3.42) obtained for the particular Fock state (4.33).

For this purpose we need to calculate the quantity

$$D^{\mathrm{cl}}_{\mu\nu}(m,n) = \langle \mathfrak{S}|D_{\mu\nu}(m,n)|\mathfrak{S}\rangle = \langle \mathfrak{S}|c^{\dagger}_{\nu}(n)c_{\mu}(m)|\mathfrak{S}\rangle. \tag{5.19}$$

Assume the system comprises finite number of Landau sites $n = 0,1,\ldots,N_L-1$ implying the limit $N_L \to \infty$ in final expressions. It is important that the algebra $W_\infty(N)$ set by (5.3) is not violated in such a finite system. For the sake of convenience we combine the isospin and site indices into a multi-index $M \equiv (\mu,m)$, which runs over the values $M = 1,2,\ldots,NN_L$. In terms of these multi-indices, the algebra $W_\infty(N)$ turns into the algebra $U(NN_L)$, and the transformation rules for fermion operators are

$$e^{-iW}c_I e^{+iW} = \sum_{I'}(U)_{II'}c_{I'}, \tag{5.20a}$$

$$e^{-iW}c^{\dagger}_I e^{+iW} = \sum_{I'}c^{\dagger}_{I'}(U^{\dagger})_{I'I}, \tag{5.20b}$$

where U is $(NN_L)\times(NN_L)$ unitary matrix

$$UU^{\dagger} = U^{\dagger}U = \mathbb{I}_{(NN_L)\times(NN_L)}. \tag{5.21}$$

Writing expectation value (5.19) in terms of multi-indices we proceed as follows

$$\begin{aligned} D^{\mathrm{cl}}_{IJ} &= \langle \mathfrak{S}|c^{\dagger}_J c_I|\mathfrak{S}\rangle = \\ &= \langle \mathfrak{S}_0|e^{-iW}c^{\dagger}_J c_I e^{+iW}|\mathfrak{S}_0\rangle = \\ &= \langle \mathfrak{S}_0|e^{-iW}c^{\dagger}_J e^{+iW}e^{-iW}c_I e^{+iW}|\mathfrak{S}_0\rangle = \\ &= \sum_{I'}\sum_{J'}(U^{\dagger})_{J'J}(U)_{II'}\langle \mathfrak{S}_0|c^{\dagger}_{J'}c_{I'}|\mathfrak{S}_0\rangle. \end{aligned} \tag{5.22}$$

In terms of multi-indices the state $|\mathfrak{S}_0\rangle$ appears as

$$|\mathfrak{S}_0\rangle = \prod_{K=1}^{NN_L}[c^{\dagger}_K]^{\nu_K}|\varnothing\rangle, \tag{5.23}$$

where from we trivially obtain

$$\langle \mathfrak{S}_0|c^{\dagger}_J c_I|\mathfrak{S}_0\rangle = \nu_I\delta_{IJ}, \tag{5.24}$$

with $\delta_{MN} \equiv \delta_{\mu\nu}\delta_{mn}$.

Substituting this into (5.22) we come to

$$D^{\mathrm{cl}}_{IJ} = \sum_K \nu_K (U)_{IK}(U^{\dagger})_{KJ}. \tag{5.25}$$

Using this together with the unitarity of U we pass further to

$$\begin{aligned} \sum_S D^{\mathrm{cl}}_{IS}D^{\mathrm{cl}}_{SJ} &= \sum_S\left[\sum_K \nu_K(U)_{IK}(U^{\dagger})_{KS}\right]\left[\sum_{K'}\nu_{K'}(U)_{SK'}(U^{\dagger})_{K'J}\right] = \\ &= \left[\sum_K \nu_K(U)_{IK}\right]\left[\sum_{K'}\nu_{K'}(U^{\dagger})_{K'J}\right]\underbrace{\sum_S(U^{\dagger})_{KS}(U)_{SK'}}_{\delta_{KK'}} = \\ &= \sum_K \nu^2_K(U)_{IK}(U^{\dagger})_{KJ}. \end{aligned} \tag{5.26}$$

Provided ν_K is either 0 or 1, we have $\nu^2_K = \nu_K$ and come to

$$\sum_S D^{\mathrm{cl}}_{IS}D^{\mathrm{cl}}_{SK} = \sum_K \nu_K(U)_{IK}(U^{\dagger})_{KJ}. \tag{5.27}$$

Comparing the right hand side to (5.25) we eventually come to

$$\sum_S D^{\rm cl}_{IS} D^{\rm cl}_{SK} = D^{\rm cl}_{IJ}. \tag{5.28}$$

Decoupling site and spin indices and taking then $N_L \to \infty$ we rewrite (5.29) as

$$\sum_{\sigma=1}^{N} \sum_{s=0}^{\infty} D^{\rm cl}_{\mu\sigma}(m,s) D^{\rm cl}_{\sigma\nu}(s,n) = D^{\rm cl}_{\mu\nu}(m,n). \tag{5.29}$$

In terms of $\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{k})$ this takes the form

$$\ell^2 \int e^{+i(\ell^2/2)\boldsymbol{k}\wedge\boldsymbol{k}'} \hat{D}^{\rm cl}_{\mu\sigma}(\boldsymbol{k}-\boldsymbol{k}') \hat{D}^{\rm cl}_{\sigma\nu}(\boldsymbol{k}') d\boldsymbol{k}' = \hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{k}), \tag{5.30}$$

where from we can pass to $\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r})$ and obtain

$$\sum_{\sigma=1}^{N} \hat{D}^{\rm cl}_{\mu\sigma}(\boldsymbol{r}) \star \hat{D}^{\rm cl}_{\sigma\nu}(\boldsymbol{r}) = \frac{1}{2\pi\ell^2}\, \hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r}). \tag{5.31}$$

This is the manifestation of non-commutativity at the level of field theory.

As we see, the non-commutativity encoded in (2.11b) becomes perceptible in terms of core fields $\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r})$, but not in terms of $D^{\rm cl}_{\mu\nu}(\boldsymbol{r})$. Relation (5.31) indicates that core fields play crucial role in studying the quantum Hall systems.

It is reasonable to rewrite (5.31) in terms of $\hat{\rho}^{\rm cl}(\boldsymbol{r})$ and $\hat{I}^{\rm cl}_a(\boldsymbol{r})$ which are related to $\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r})$ as

$$\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r}) = \frac{1}{N}\,\delta_{\mu\nu}\,\hat{\rho}^{\rm cl}(\boldsymbol{r}) + (\lambda_a)_{\mu\nu}\hat{I}^{\rm cl}_a(\boldsymbol{r}). \tag{5.32}$$

Substituting this into (5.31) and using the properties of Gell-Mann matrices we obtain

$$\hat{I}^{\rm cl}_a \star \hat{I}^{\rm cl}_a = \frac{1}{4\pi\ell^2}\,\hat{\rho}^{\rm cl} - \frac{1}{2N}\,\hat{\rho}^{\rm cl} \star \hat{\rho}^{\rm cl}, \tag{5.33a}$$

$$\big(if_{abc} + d_{abc}\big)\hat{I}^{\rm cl}_b \star \hat{I}^{\rm cl}_c = \frac{1}{2\pi\ell^2}\,\hat{I}^{\rm cl}_a - \frac{1}{N}\,\hat{\rho}^{\rm cl} \star \hat{I}^{\rm cl}_a - \frac{1}{N}\,\hat{I}^{\rm cl}_a \star \hat{\rho}^{\rm cl}. \tag{5.33b}$$

In order to point out the essence of these relations we first study the homogeneous configuration. So far we have not concerned the dynamical aspects. We shall comment on this issue only in the subsequent lectures. Here we only state that homogeneous configurations arise as ground states of dynamical models.

In the ground state the filling of a separate site $\nu(m)$ takes one and the same integer value for all m. This value is the value of the filling factor given by (5.18), *i.e.* we have $\nu(m) = \nu$ where ν is an integer with $0 \leqslant \nu \leqslant N$. Consequently, we have

$$\hat{\rho}^{\rm cl}(\boldsymbol{r}) = \frac{\nu}{2\pi\ell^2}. \tag{5.34a}$$

For the spin field we employ the similar normalization and write

$$\hat{I}^{\rm cl}_a(\boldsymbol{r}) = \frac{\mathcal{I}_a}{2\pi\ell^2}. \tag{5.34b}$$

Remind now that for homogeneous configurations the NC product coincides with the regular (commutative) one. Therefore, substituting (5.34) into (5.33) we come to

$$\mathcal{I}_a \mathcal{I}_a = \frac{\nu(N-\nu)}{2N}, \tag{5.35a}$$

$$d_{abc}\,\mathcal{I}_b \mathcal{I}_c = \frac{N-2\nu}{N}\,\mathcal{I}_a, \tag{5.35b}$$

where ν is an integer with $0 \leqslant \nu \leqslant N$.

Relations (5.35) define the so-called Grassmannian manifold $G_{N,\nu}$. In this connection, we regard the relations (5.33) as a definition of the non-commutative Grassmannian manifold.

5.3 Commutative Grassmannian Fields

Non-commutative relations (5.31) and (5.33) can be resolved in terms of non-commutative Grassmannian fields. Though (5.33) look more physical, the form (5.31) is more compact and convenient to work with.

Before resolving this relation in terms of non-commutative Grassmannian fields, it would be instructive first to get acquainted with commutative Grassmannian fields.

Introduce the N-component column

$$f(\boldsymbol{r}) = \begin{pmatrix} f_1(\boldsymbol{r}) \\ \vdots \\ f_N(\boldsymbol{r}) \end{pmatrix}, \tag{5.36}$$

where $f_\sigma(\boldsymbol{r})$ are complex-valued functions subject to the normalization

$$f^\dagger(\boldsymbol{r}) f(\boldsymbol{r}) = \sum_{\sigma=1}^{N} f_\sigma^*(\boldsymbol{r}) f_\sigma(\boldsymbol{r}) = 1. \tag{5.37}$$

Then the quantities

$$\mathcal{D}_{\mu\nu}(\boldsymbol{r}) \equiv \frac{1}{2\pi\ell^2} f_\mu(\boldsymbol{r}) f_\nu^*(\boldsymbol{r}) \tag{5.38}$$

satisfy the relations

$$\mathcal{D}_{\mu\sigma}(\boldsymbol{r})\mathcal{D}_{\sigma\nu}(\boldsymbol{r}) = \frac{1}{2\pi\ell^2}\mathcal{D}_{\mu\nu}(\boldsymbol{r}), \tag{5.39}$$

which are the commutative counterparts of (5.31).

Calculating the particle and spin densities we find

$$\varrho(\boldsymbol{r}) = \mathrm{Tr}\big[\mathcal{D}(\boldsymbol{r})\big] = \rho_L, \tag{5.40a}$$

$$\mathcal{I}_a(\boldsymbol{r}) = \tfrac{1}{2}\mathrm{Tr}\big[\lambda_a \mathcal{D}(\boldsymbol{r})\big] = \tfrac{1}{2}\rho_L f^\dagger(\boldsymbol{r})\lambda_a f(\boldsymbol{r}), \tag{5.40b}$$

i.e. these describe isospin field configurations with homogeneous particle density ρ_L.

Among the variety of $f(\boldsymbol{r})$ there are ones which differ from each other by an overall phase factor. These copies lead to identical $\mathcal{D}_{\mu\sigma}(\boldsymbol{r})$, and we identify them as describing one and the same physical configuration. Factorizing the space of $f(\boldsymbol{r})$ with this equivalence we obtain what is referred to as the complex projective space with its elements being the complex projective fields, or the CP^{N-1} fields, in short. The N-component complex-valued column comprises $2N$ real parameters, one of which is fixed by normalization (5.37), and another one is fixed after removing the overall phase. Therefore, the CP^{N-1} field comprises $2(N-1)$ real parameters.

The CP^{N-1} field represents the particular case of Grasmmannian field $G_{N,\nu}$ where $\nu = 1$. For defining $G_{N,\nu}$ let us introduce ν amount of columns

$$f^1 = \begin{pmatrix} f_1^1 \\ f_2^1 \\ \vdots \\ f_N^1 \end{pmatrix}, \qquad f^2 = \begin{pmatrix} f_1^2 \\ f_2^2 \\ \vdots \\ f_N^2 \end{pmatrix}, \qquad \cdots \qquad f^\nu = \begin{pmatrix} f_1^\nu \\ f_2^\nu \\ \vdots \\ f_N^\nu \end{pmatrix}, \tag{5.41}$$

subject to

$$[f^r(\boldsymbol{r})]^\dagger f^s(\boldsymbol{r}) = \sum_{\sigma=1}^{N} [f_\sigma^r(\boldsymbol{r})]^* f_\sigma^s(\boldsymbol{r}) = \delta^{rs}. \tag{5.42}$$

Then the quantity

$$\mathcal{D}_{\mu\nu}(\boldsymbol{r}) \equiv \frac{1}{2\pi\ell^2} \sum_{s=1}^{\nu} f_\mu^s(\boldsymbol{r})[f_\nu^s(\boldsymbol{r})]^* \tag{5.43}$$

satisfies the same equation (5.39). The corresponding particle and spin densities are

$$\varrho(\boldsymbol{r}) = \mathrm{Tr}\big[\mathcal{D}(\boldsymbol{r})\big] = \rho_L \nu, \tag{5.44a}$$

$$\mathcal{I}_a(\boldsymbol{r}) = \tfrac{1}{2}\mathrm{Tr}\big[\lambda_a \mathcal{D}(\boldsymbol{r})\big] = \tfrac{1}{2}\rho_L \sum_{s=1}^{\nu} [f^s(\boldsymbol{r})]^\dagger \lambda_a f^s(\boldsymbol{r}), \tag{5.44b}$$

where the particle number density is ν-times more compared to (5.40a).

Like in the case of (5.38), the quantity set by (5.43) is invariant under the transformation

$$f^r_\mu(\boldsymbol{r}) \to U^{rs} f^s_\mu(\boldsymbol{r}) \qquad\qquad U^\dagger U = \mathbb{I}_{\nu\times\nu}. \tag{5.45}$$

The space of $\{f^1, f^2, \ldots, f^\nu\}$ factorised with respect to this equivalence is the Grassmannian manifold $G_{N,\nu}$. Number of real free parameters comprised in ν amount of N-component complex-valued columns is $2N\nu$. Normalizations (5.42) fix ν real parameters for $r = s$, and $\nu^2 - \nu$ real ones for $r \neq s$, *i.e.* ν^2 real parameters in total. Removing the freedom associated with U-transformation (5.45) we fix more ν^2 real parameters. Hence the Grassmannian field $G_{N,\nu}$ comprises $2\nu(N-\nu)$ real degrees of freedom.

5.4 Non-Commutative Grassmannian Fields

We now turn to NC Grassmannian fields. Analogously to the commutative case here we have $G_{N,1} = CP^{N-1}$. Therefore we do not spread on considering NC CP^{N-1} separately but comment right on NC $G_{N,\nu}$.

Again, we introduce

$$f^1 = \begin{pmatrix} f^1_1 \\ f^1_2 \\ \vdots \\ f^1_N \end{pmatrix}, \qquad f^2 = \begin{pmatrix} f^2_1 \\ f^2_2 \\ \vdots \\ f^2_N \end{pmatrix}, \qquad \cdots \qquad f^\nu = \begin{pmatrix} f^\nu_1 \\ f^\nu_2 \\ \vdots \\ f^\nu_N \end{pmatrix}, \tag{5.46}$$

which in contrast to (5.42) are subject to

$$[f^r(\boldsymbol{r})]^\dagger \star f^s(\boldsymbol{r}) = \sum_{\sigma=1}^{N} [f^r_\sigma(\boldsymbol{r})]^* \star f^s_\sigma(\boldsymbol{r}) = \delta^{rs}. \tag{5.47}$$

Then the quantities $\hat{D}^{\mathrm{cl}}_{\mu\nu}(\boldsymbol{r})$ satisfying the non-commutative relation (5.31) can be written as

$$\hat{D}^{\mathrm{cl}}_{\mu\nu}(\boldsymbol{r}) = \frac{1}{2\pi\ell^2} \sum_{s=1}^{\nu} f^s_\mu(\boldsymbol{r}) \star [f^s_\nu(\boldsymbol{r})]^*. \tag{5.48}$$

Remind that $f_1 \star f_2 \neq f_2 \star f_1$, in general, so the order of multiplicatives is a sensitive case in (5.47) and (5.48). Particle and spin densities appear as

$$\hat{\rho}^{\mathrm{cl}}(\boldsymbol{r}) = \mathrm{Tr}\big[\hat{D}^{\mathrm{cl}}(\boldsymbol{r})\big] = \rho_L \sum_{s=1}^{\nu} f^s_\mu(\boldsymbol{r}) \star [f^s_\mu(\boldsymbol{r})]^*, \tag{5.49a}$$

$$\hat{I}^{\mathrm{cl}}_a(\boldsymbol{r}) = \tfrac{1}{2}\mathrm{Tr}\big[\lambda_a \hat{D}^{\mathrm{cl}}(\boldsymbol{r})\big] = \rho_L \sum_{s=1}^{\nu} (\tfrac{1}{2}\lambda_a)_{\nu\mu} f^s_\mu(\boldsymbol{r}) \star [f^s_\nu(\boldsymbol{r})]^*. \tag{5.49b}$$

In contrast to commutative case we cannot interchange the places of $f^s_\mu(\boldsymbol{r})$ and $[f^s_\mu(\boldsymbol{r})]^*$ in (5.49a)

$$f^s_\mu(\boldsymbol{r}) \star [f^s_\mu(\boldsymbol{r})]^* \neq [f^s_\mu(\boldsymbol{r})]^* \star f^s_\mu(\boldsymbol{r}), \tag{5.50}$$

and therefore the particle density is not homogeneous any more, but is spatially modulated.

Like in commutative case, here we also have various sets $\{f^1, f^2, \ldots, f^\nu\}$ producing identical particle and spin densities. Indeed, let $U(\boldsymbol{r})$ be a $\nu \times \nu$ non-commutative unitary matrix

$$\begin{aligned} U(\boldsymbol{r}) \star U^\dagger(\boldsymbol{r}) = \mathbb{I}_{\nu\times\nu} \qquad &\Rightarrow \qquad U^{us}(\boldsymbol{r}) \star [U^{vs}(\boldsymbol{r})]^* = \delta^{uv}, \\ U(\boldsymbol{r})^\dagger \star U(\boldsymbol{r}) = \mathbb{I}_{\nu\times\nu} \qquad &\Rightarrow \qquad [U^{ur}(\boldsymbol{r})]^* \star U^{us}(\boldsymbol{r}) = \delta^{rs}, \end{aligned} \tag{5.51}$$

and consider the transformation

$$f^s_\sigma(\boldsymbol{r}) \;\to\; f^u_\sigma(\boldsymbol{r}) \star U^{us}(\boldsymbol{r}), \tag{5.52a}$$

$$[f^s_\sigma(\boldsymbol{r})]^* \;\to\; [U^{us}(\boldsymbol{r})]^* \star [f^u_\sigma(\boldsymbol{r})]^*. \tag{5.52b}$$

Remark that these two relations follow one from another by use of the property $(f \star h)^* = h^* \star f^*$. Using (5.51) we find that (5.52) maintains the normalization (5.46) while (5.48) is invariant. Factorizing thew space of $\{f^1, f^2, \ldots, f^\nu\}$ with respect to this equivalence we obtain the space of non-commutative Grassmannian fields.

It is instructive to bring the example of the simplest ($\nu = 1$) case of (5.51). This is the non-commutative exponent

$$\mathfrak{e}^{i\xi(\boldsymbol{r})} \equiv 1 + \frac{i}{1!}\xi + \frac{i^2}{2!}\xi \star \xi + \frac{i^3}{3!}\xi \star \xi \star \xi + \cdots \tag{5.53}$$

which obviously satisfies $\mathfrak{e}^{i\xi(\boldsymbol{r})} \star \mathfrak{e}^{-i\xi(\boldsymbol{r})} = 1$.

Problems

(5.a) Show that hermiticity of W set by (5.10) requires (5.11).

(5.b) Employing the anticommutation relation derive (5.20) for $NN_L = 2$.

(5.c) Derive (5.29) → (5.30) → (5.31).

(5.d) Use the properties (4.15) of "Gell-Mann" matrices and obtain (5.33) from (5.31).

(5.e) Show that (5.47) and (5.48) are invariant against the non-commutative transformations (5.51).

Chapter 6

TOPOLOGICAL CHARGE OF NON-COMMUTATIVE GRASSMANNIAN FIELDS

We comment on topological characteristics of non-commutative Grassmannian fields. For this purpose we first briefly present the basics of homotopy and topology.

6.1 Basic Aspects of Homotopy Theory

Consider a plane with a hole as shown in figure 6.1, and the closed curves in it starting and ending at point x_0. The curves shown in left panel can be *continuously* deformed one into another, while the ones in middle panel cannot be deformed *continuously* one into another due to the hole. In this light we introduce the notion of equivalence: if any two curves can be *continuously* deformed one into another, we say they are homotopically equivalent, hence belong to one and the same homotopy class, otherwise to different classes. For example, the curves A and B belong to the same class, while C is from another class. Homotopy classes can be labelled by an integer number counting the times curves encircle the hole, *e.g.* the curve C encircles the hole once, the curve D in the right panel encircles it twice; A and B do not encircle at all, and so on.

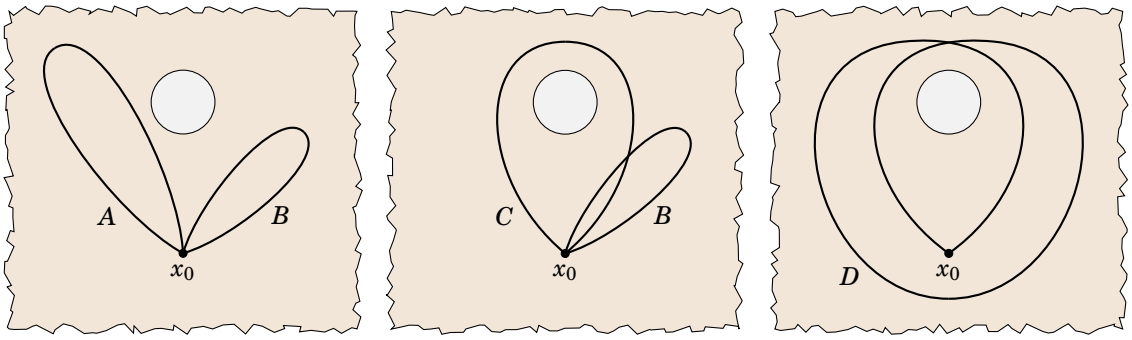


Fig. 6.1:

Each curve can be endowed with direction it encircles the hole, For example, the curve C can be redrawn in two different ways as shown in figure 6.2. Hence the left curve is assigned the integer "+1" (anticlockwise is positive direction) while the one on the right the integer "−1" (clockwise is negative direction). The curves like D encircling the hole twice may carry the number either "+2" or "−2", and so on.

The essence of direction is absent for the curves like A and B: even if we endow them by any direction, we can shrink them to a point and then restore back with opposite direction. Such a trick cannot be done for the two curves shown in figure 6.2 because they cannot be shrunk to a point x_0. In this scope the curves labelled by opposite numbers are attributed to different homotopy classes.

We have thus broken apart the set of curves into homotopy classes labelled by integer numbers. Remind in this connection that integers form the group denoted by $\mathbb{Z}$ with respect to addition. Correspondingly, the set of homotopy classes can be endowed by the analogous group structure. Let $\bar{C}_{+1}$ and $\tilde{C}_{+1}$ be two loops starting and ending at the same point x_0 and both encircling the hole once in positive direction as shown in figure 6.3. Continuously detaching the start point of $\bar{C}_{+1}$ together with the end point of $\tilde{C}_{+1}$ away from x_0 we pass to what is shown in middle panel and further to C_{+2} shown in right panel. We express this fact as $\bar{C}_{+1} + \tilde{C}_{+1} = C_{+2}$. Thus, composition of a curve labeled by m with another one labeled by n produces the curve with $m+n$. In this view the homotopy classes form the group which is denoted as $\pi_1(M)$ and is referred to as the *fundamental group*. The subscript "1" indicates the involved objects (loops) over the space M are one-dimensional. In the given case this is an Abelian group isomorphic to the one of integers and we express this fact as $\pi_1(P) = \mathbb{Z}$.

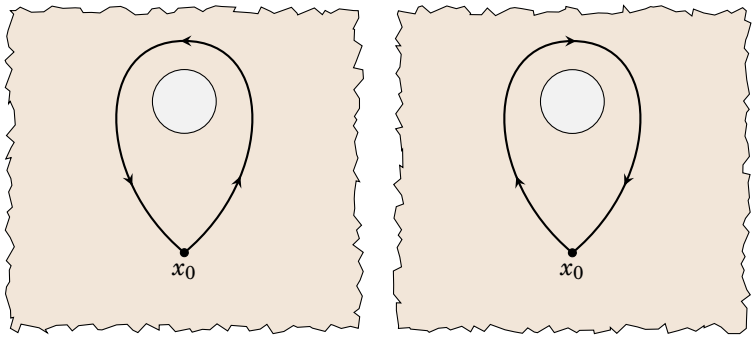


Fig. 6.2:

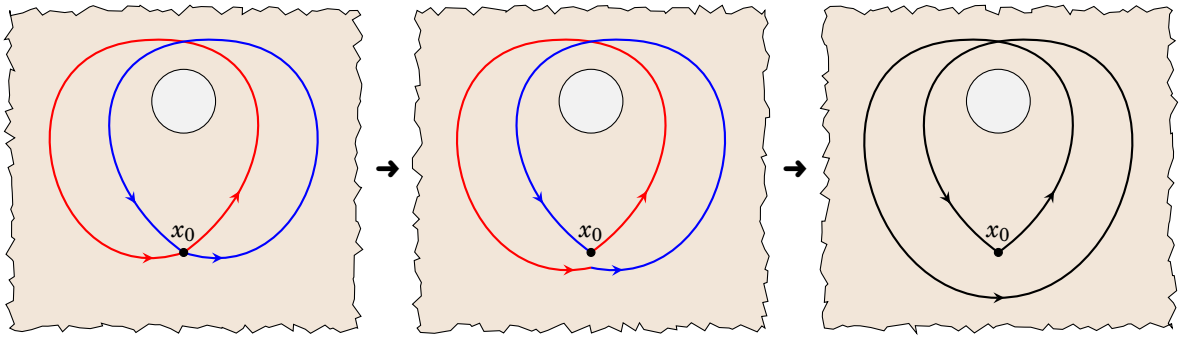


Fig. 6.3:

The same homotopy group arises when considering the mapping from S^1 to another S^1. Such structures arise in certain physical theories where the phase fields are defined over the xy-plane. Parameterizing the points of a plane as $re^{i\phi}$, the phase field can be written as $A(r,\phi) = e^{if(r,\phi)}$ and must be single-valued, *i.e.* performing one turn in the plane around the origin (ϕ varies from some value ϕ_0 to $\phi_0 + 2\pi$) the phase field must take the same value. Therefore we must have $f(r,\phi+2\pi) = f(r,\phi)+2\pi\mathbb{Z}$. As an explicit realizations we can bring $f_n(r,\phi) = n\phi$ with n being an integer counting the times one S^1 winds around the other. Analytically this number can be expressed as

$$n = \frac{1}{2\pi i}\oint A^* \partial_\phi A d\phi. \tag{6.1}$$

Multiplication of two such functions is given by $A_m A_n = A_{m+n}$ and gives rise to the group structure which is expressed as $\pi_1(S^1) = \mathbb{Z}$.

Considering loops on S^2 we come to trivial homotopy group: any closed loop on a sphere is contractible to a point hence we have $\pi_1(S^2) = 0$.

We bring yet another example: loops on a torus T^2. In that case a loop may encircle the major circle as well as the minor one, as shown in figure 6.4. Obviously, a loop may simultaneously encircle m times one of these circles, and n times the other. Therefore, every loop on T^2 is characterized by two integers, hence the homotopy group is the direct product of two $\mathbb{Z}$ groups: $\pi_2(T^2) = \mathbb{Z}\otimes\mathbb{Z}$.

Fundamental group reflects the properties of loops under continuous deformations hence characterizes topological properties of the space the loops are considered in: loops on a plane feel the presence of a hole because the ones with distinct winding numbers cannot be continuously deformed one into another. In contrast, loops in three-dimensional space are all continuously contractible to a point even if any points of a space are removed; in other words there is no sense of encircling a point by a loop in three dimensions. Therefore, the fundamental group is trivial hence incapable of detecting such defects in three dimensions. However, such detection is possible if we employ 2-spheres instead of loops what gives rise to the homotopy group $\pi_2(M)$. Higher homotopy groups $\pi_n(M)$ can be introduced by considering n-spheres embedded in M.

Below we comment on $\pi_2(S^2)$ which is directly related to skyrmions and Grassmannian fields arising in the context of the field theory in the lowest Landau level.

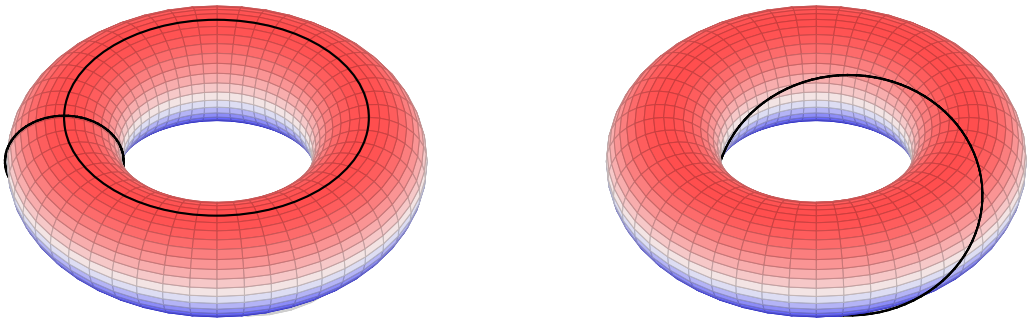


Fig. 6.4: Mapping from a sphere S^1 to torus T^2: left panel shows $(m,n)=(1,0)$ and $(m,n)=(0,1)$; right panels shows $(m,n)=(1,1)$.

6.2 Non-Linear Sigma Model and Commutative Winding Number

Consider the field $\mathcal{J}=(\mathcal{J}_x,\mathcal{J}_y,\mathcal{J}_z)$ defined on a plane and subject to the constraint

$$\mathcal{J}^2=\sum_a \mathcal{J}_a^2=1. \tag{6.2}$$

Non-linear sigma model is defined by the following energy functional

$$E=\tfrac{1}{2}\int(\partial_k\mathcal{J}_a)^2 d\boldsymbol{r}. \tag{6.3}$$

In order the energy to be finite the derivatives $\partial_k\mathcal{J}_a$ must vanish at spatial infinity with sufficient velocity. This implies the field $\mathcal{J}$ must approach a constant value as $r\to\infty$. Without loss of generality we may assume this constant value is the "north pole" $\mathcal{J}=(0,0,1)$. This is shown in figure 6.5.

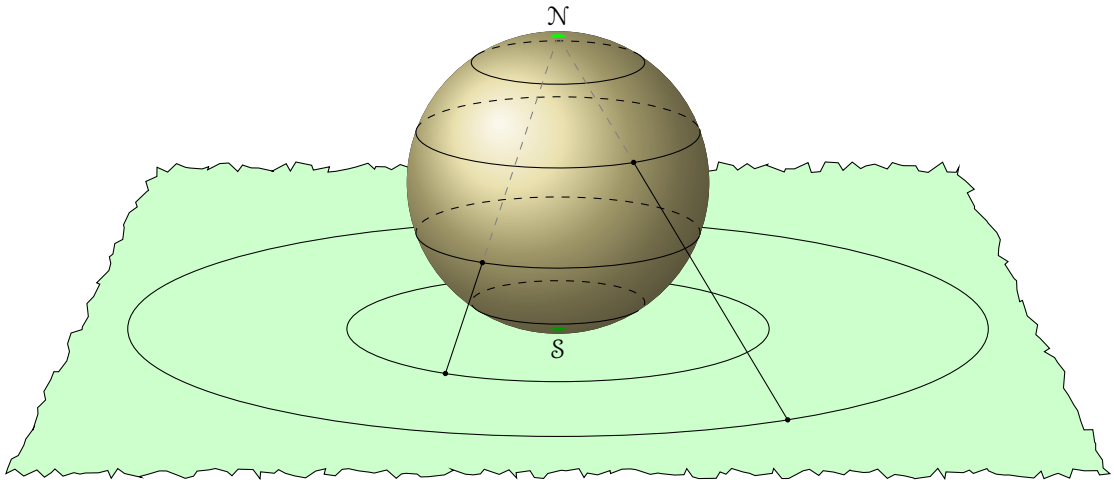


Fig. 6.5: Mapping $R^2\to S^2$.

All points of an infinite circle ($r=\infty$) are mapped into a single point $\mathcal{J}=(0,0,1)$. In such a cases we say that the original R^2 is compactified into S^2. Hence instead of mapping $R^2\to S^2$ we have the one $S^2\to S^2$. Such mappings are classified in accord with the homotopy group $\pi_2(S^2)=\mathbb{Z}$, where the integer counts the times the field space S^2 is wrapped when the coordinate $\boldsymbol{r}$ spans the entire plane. Such configurations are called $O(3)$ skyrmions and the corresponding winding number (topological charge, Pontryagin number) is given by

$$Q=\frac{1}{8\pi}\int\epsilon_{abc}\epsilon_{ij}\mathcal{J}_a(\partial_i\mathcal{J}_b)(\partial_j\mathcal{J}_c)d\boldsymbol{r}. \tag{6.4}$$

We proceed to explicitly construct skyrmion configurations by minimizing the energy functional (6.3). For this purpose we consider the obvious inequality

$$\int (\partial_i \mathcal{J}_a \pm \epsilon_{ij}\epsilon_{abc}\mathcal{J}_b\partial_j\mathcal{J}_c)^2 d\boldsymbol{r} \geqslant 0. \tag{6.5}$$

Due to $\mathcal{J}_a\mathcal{J}_a = 1$ we have $\mathcal{J}_a\partial_j\mathcal{J}_a = 0$ hence the inequality can be rewritten as

$$\int (\partial_i\mathcal{J}_a)^2 d\boldsymbol{r} \geqslant \pm \int \epsilon_{ij}\epsilon_{abc}\mathcal{J}_a(\partial_i\mathcal{J}_b)(\partial_j\mathcal{J}_c)d\boldsymbol{r}. \tag{6.6}$$

This is equivalent to

$$E \geqslant 4\pi Q, \tag{6.7}$$

where E is the energy of a configuration, and Q is its winding number.

Energy becomes minimal when the equality is held in (6.7) and correspondingly in (6.5). This leads to

$$\partial_i\mathcal{J}_a \pm \epsilon_{ij}\epsilon_{abc}\mathcal{J}_b\partial_j\mathcal{J}_c = 0, \tag{6.8}$$

and such configurations are referred to as the Bogomol'nyi-Prasad-Sommerfeld (BPS) solitons.

Details on solving this equation can be found *e.g.* in section 7.7 of *Ezawa*, while here we directly bring the typical skyrmion solutions. These are given by

$$\mathcal{J}_x = \frac{2\kappa^n r^n}{r^{2n}+\kappa^{2n}}\cos(n\phi), \qquad \mathcal{J}_y = \frac{-2\kappa^n r^n}{r^{2n}+\kappa^{2n}}\sin(n\phi), \qquad \mathcal{J}_z = \frac{r^{2n}-\kappa^{2n}}{r^{2n}+\kappa^{2n}}, \tag{6.9}$$

where κ is a positive constant specifying the scale of a skyrmion, and q is positive integer. Calculating the winding number (6.4) we find $Q = n$.

Another solution named as antiskyrmion is given by

$$\mathcal{J}_x = \frac{2\kappa^n r^n}{r^{2n}+\kappa^{2n}}\cos(n\phi), \qquad \mathcal{J}_y = \frac{2\kappa^n r^n}{r^{2n}+\kappa^{2n}}\sin(n\phi), \qquad \mathcal{J}_z = \frac{r^{2n}-\kappa^{2n}}{r^{2n}+\kappa^{2n}}, \tag{6.10}$$

and carries the winding number $Q = -n$.

In section 5.3 we introduced the commutative CP^{N-1} field by (5.37) and the corresponding isospin field by (5.40b). For $N = 2$ those constructions give the sigma-field $(\mathcal{J}_x, \mathcal{J}_y, \mathcal{J}_z)$ we have just considered here. We now proceed to express the winding number (6.4) of a sigma-field in terms of the underlying CP^1 field $f(\boldsymbol{r})$. For this purpose we omit the dimensional factor of ρ_L and write (5.40b) as

$$\mathcal{J}_a(\boldsymbol{r}) = \tfrac{1}{2} f^\dagger(\boldsymbol{r})\sigma_a f(\boldsymbol{r}), \tag{6.11}$$

where σ_a are the Pauli matrices. Substituting this into (6.4) we find

$$Q = \frac{1}{2\pi i}\epsilon_{kl}\sum_{\mu=1}^{N}\int (\partial_k \bar{f}_\mu)(\partial_l f_\mu)d\boldsymbol{r} \tag{6.12}$$

where $N = 2$.

Compared to (6.4), this expression is more convenient because it provides with the straightforward way for generalizing the winding number expression first to CP^{N-1} fields, then to Grassmannian fields $G_{N,\nu}$, and finally to their non-commutative versions. In particular, the winding number of a CP^{N-1} field is given by the same expression (6.12) where $f(\boldsymbol{r})$ is the N-component column.

As introduced in section 5.3 the Grassmannian field $G_{N,\nu}$ is formed by $f^1(\boldsymbol{r}), f^2(\boldsymbol{r}), \ldots, f^\nu(\boldsymbol{r})$, each representing an N-component column and satisfying (5.42). In that case the expression (6.12) is generalized to

$$Q = \frac{1}{2\pi i}\epsilon_{kl}\sum_{\mu=1}^{N}\sum_{s=1}^{\nu}\int (\partial_k \bar{f}^s_\mu)(\partial_l f^s_\mu)d\boldsymbol{r}, \tag{6.13}$$

which allows the further straightforward generalization to non-commutative Grassmannian fields.

6.3 Non-Commutative Winding Number

The prior requirement while generalizing the winding number to non-commutative case is that in the commutative limit it must be reduced back to (6.13). Remind, now that the expansion (3.2) for the NC product can be expressed as

$$f_1 \star f_2 = f_1 f_2 - \tfrac{i}{2}\theta\epsilon_{nj}(\partial_n f_1)(\partial_j f_2) + \mathcal{O}(\theta^2). \tag{6.14}$$

Then, introducing the so-called *Moyal bracket* as

$$\big[f_1, f_2\big]_\star = f_1 \star f_2 - f_2 \star f_1 \tag{6.15}$$

we can write

$$\big[\bar{f}^s_\mu(\boldsymbol{r}), f^s_\mu(\boldsymbol{r})\big]_\star = -i\theta\epsilon_{ij}(\partial_i \bar{f}^s_\mu)(\partial_j f^s_\mu) + \mathcal{O}(\theta^2). \tag{6.16}$$

This expression suggests the natural NC generalization of (6.13)

$$Q = \frac{1}{2\pi\theta}\sum_{\mu=1}^{N}\sum_{s=1}^{\nu}\int \big[\bar{f}^s_\mu(\boldsymbol{r}), f^s_\mu(\boldsymbol{r})\big]_\star \, d\boldsymbol{r}. \tag{6.17}$$

Topological charge can be expressed in a variety of ways and they are all equivalent. The corresponding topological charge densities are distinct and carry no physical meaning, in general. Nevertheless the one corresponding to (6.17) is preferred since it is related to electron density excitation. Indeed, the topological charge density for (6.17) looks as

$$\mathcal{Q}(\boldsymbol{r}) = \frac{1}{2\pi\theta}\sum_{\mu=1}^{N}\sum_{s=1}^{\nu}\big[\bar{f}^s_\mu(\boldsymbol{r}), f^s_\mu(\boldsymbol{r})\big]_\star =$$

$$= \frac{1}{2\pi\theta}\sum_{\mu=1}^{N}\sum_{s=1}^{\nu}\bar{f}^s_\mu(\boldsymbol{r}) \star f^s_\mu(\boldsymbol{r}) - \frac{1}{2\pi\theta}\sum_{\mu=1}^{N}\sum_{s=1}^{\nu} f^s_\mu(\boldsymbol{r}) \star \bar{f}^s_\mu(\boldsymbol{r}). \tag{6.18}$$

Taking into account (5.47) and (5.49a) we rewrite this as

$$\mathcal{Q}(\boldsymbol{r}) = -\big[\hat{\rho}_{\mathrm{cl}}(\boldsymbol{r}) - \nu\rho_L\big]. \tag{6.19}$$

Later on we demonstrate that the quantity $\nu\rho_L$ represents the ground state value of the particle density. Consequently, the combination $\hat{\rho}_{\mathrm{cl}}(\boldsymbol{r}) - \nu\rho_L$ describes the excitation part of the charge density and we have

$$Q = -\Delta N_e^{\mathrm{cl}} \tag{6.20}$$

where ΔN_e^{cl} is the number of extra particles brought by excitations. Remark, that the states defined by (5.15) and (5.16) comprise definite amount of particles. Once the ground and excited states both fall in this category, the quantity ΔN_e^{cl} is always integer, so the topological charge is also integer.

Summarizing, non-commutative soliton necessarily carries the proper electric charge proportional to its topological charge. This is the peculiarity of non-commutative theories which has no analogue in commutative constructions.

Problems

(6.a) Calculate the topological charge for skyrmions (6.9) and antiskyrmions (6.10).

(6.b) Derive (6.12) from (6.4).

Chapter 7

ELECTRON-ELECTRON INTERACTIONS

We apply the second quantization formalism for describing four-fermion interactions in the lowest Landau level. We start from deriving the direct and exchange interaction energies. The later is shown to comprise the non-commutative CP-models as its low-energy constituent.

We show that in the ground state the isospin averages are spontaneously aligned along an arbitrarily chosen direction. This takes place due to the four-fermion exchange interaction even in the absence of the Zeeman term. For that reason the system is called the quantum Hall ferromagnet.

We then study excitations over the ground state. For the sake of simplicity we consider the case of $N = 2$ ($\mu = \uparrow, \downarrow$) where the filling $\nu = 1$ is the only nontrivial one.

7.1 Four-Fermion Interaction

As we have already pointed out, the kinetic Hamiltonian is quenched in the lowest Landau level, hence the dynamics is governed solely by the electron-electron (four-fermion) interactions. The later is described by the second quantized Hamiltonian

$$H_e = \tfrac{1}{2}\int \big[\rho(\boldsymbol{r}_1) - \rho_e\big] V(\boldsymbol{r}_1 - \boldsymbol{r}_2) \big[\rho(\boldsymbol{r}_2) - \rho_e\big] d\boldsymbol{r}_1 d\boldsymbol{r}_2, \tag{7.1}$$

where $\rho(\boldsymbol{r})$ is the particle density operator, and $e\rho_e$ is the density of constrained charges providing electric neutrality of the system in the ground state.

One-particle potential $V(\boldsymbol{r})$ is of Coulomb type ($\sim r^{-1}$) in real systems, but here we consider the general case with the only assumption that $V(\boldsymbol{r})$ is repulsive.

In what follows we calculate the average value $E \equiv \langle \mathfrak{S}|H_e|\mathfrak{S}\rangle$ and express it in terms of core fields $\hat{\rho}_{\rm cl} \equiv \langle \mathfrak{S}|\hat{\rho}|\mathfrak{S}\rangle$ and $\hat{I}_a^{\rm cl} \equiv \langle \mathfrak{S}|\hat{I}_a|\mathfrak{S}\rangle$. Here $|\mathfrak{S}\rangle$ is the Fock state set by (5.15) and (5.16) which (as previously mentioned) embraces all physically relevant configurations as ground state and various excited states. Sandwiching (7.1) we find

$$\begin{aligned}
\langle \mathfrak{S}|H_e|\mathfrak{S}\rangle = {} & \tfrac{1}{2}\int V(\boldsymbol{r}_1 - \boldsymbol{r}_2)\langle \mathfrak{S}|\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)|\mathfrak{S}\rangle d\boldsymbol{r}_1 d\boldsymbol{r}_2 - \\
& - \tfrac{1}{2}\rho_e \int V(\boldsymbol{r}_1 - \boldsymbol{r}_2)\langle \mathfrak{S}|\rho(\boldsymbol{r}_2)|\mathfrak{S}\rangle d\boldsymbol{r}_1 d\boldsymbol{r}_2 - \\
& - \tfrac{1}{2}\rho_e \int V(\boldsymbol{r}_1 - \boldsymbol{r}_2)\langle \mathfrak{S}|\rho(\boldsymbol{r}_1)|\mathfrak{S}\rangle d\boldsymbol{r}_1 d\boldsymbol{r}_2 + \\
& + \tfrac{1}{2}\rho_e^2 \int V(\boldsymbol{r}_1 - \boldsymbol{r}_2)\langle \mathfrak{S}|\mathfrak{S}\rangle d\boldsymbol{r}_1 d\boldsymbol{r}_2.
\end{aligned} \tag{7.2}$$

In the last term of this expression we employ the normalization $\langle \mathfrak{S}|\mathfrak{S}\rangle = 1$. The two middle ones are calculable as

$$\int V(\boldsymbol{r}_1 - \boldsymbol{r}_2)\langle \mathfrak{S}|\rho(\boldsymbol{r}_2)|\mathfrak{S}\rangle d\boldsymbol{r}_1 d\boldsymbol{r}_2 = \int V(\boldsymbol{r}_1 - \boldsymbol{r}_2)\rho_{\rm cl}(\boldsymbol{r}_2) d\boldsymbol{r}_1 d\boldsymbol{r}_2. \tag{7.3}$$

We now pass to calculating the first term of (7.2). For this purpose we first express it in terms of creation and annihilation operators. Using expansion (3.1) in $\rho = \psi^\dagger\psi$ we find

$$\rho(\boldsymbol{r}) = \sum_{mn} \langle 0, m|\boldsymbol{r}\rangle\langle \boldsymbol{r}|0, n\rangle c_\mu^\dagger(m) c_\mu(n). \tag{7.4}$$

Substituting this into the first term of (7.2) takes the form

$$\tfrac{1}{2}\int V(\boldsymbol{r}_1-\boldsymbol{r}_2)\langle\mathfrak{S}|\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)|\mathfrak{S}\rangle d\boldsymbol{r}_1 d\boldsymbol{r}_2 = \sum V_{mnst}\langle\mathfrak{S}|c^\dagger_\mu(m)c_\mu(n)c^\dagger_\nu(s)c_\nu(t)|\mathfrak{S}\rangle, \tag{7.5}$$

where we have introduce the notation

$$V_{mnij} \equiv \tfrac{1}{2}\int V(\boldsymbol{r}_1-\boldsymbol{r}_2)\langle 0,m|\boldsymbol{r}_1\rangle\langle\boldsymbol{r}_1|0,n\rangle\langle 0,s|\boldsymbol{r}_2\rangle\langle\boldsymbol{r}_2|0,t\rangle d\boldsymbol{r}_1 d\boldsymbol{r}_2. \tag{7.6}$$

Using the Fourier transform

$$V(\boldsymbol{r}_1-\boldsymbol{r}_2) = \frac{1}{2\pi}\int V(\boldsymbol{k})e^{+i\boldsymbol{k}(\boldsymbol{r}_1-\boldsymbol{r}_2)}d\boldsymbol{k}, \tag{7.7}$$

we bring this to

$$V_{mnst} = \frac{1}{4\pi}\int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})\langle m|e^{+i\boldsymbol{k}\boldsymbol{X}}|n\rangle\langle s|e^{-i\boldsymbol{k}\boldsymbol{X}}|t\rangle d\boldsymbol{k}. \tag{7.8}$$

Later we will employ the properties of V_{mnst} which trivially follow from (7.8). These are

$$\sum_n V_{mnnt} = \frac{1}{4\pi}\delta_{mt}\int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})d\boldsymbol{k} \tag{7.9}$$

and $V_{mnst} = V_{stmn}$.

7.2 Calculating Four-Point Averages

We present the details of calculating the four-point average values $\langle\mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)c^\dagger_\sigma(s)c_\tau(t)|\mathfrak{S}\rangle$. Using the anticommutation relations (4.2) we write

$$\langle\mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)c^\dagger_\sigma(s)c_\tau(t)|\mathfrak{S}\rangle = \langle\mathfrak{S}|c^\dagger_\mu(m)\Big[\delta_{\sigma\nu}\delta_{sn} - c^\dagger_\sigma(s)c_\nu(n)\Big]c_\tau(t)|\mathfrak{S}\rangle =$$

$$= \delta_{\sigma\nu}\delta_{sn}\langle\mathfrak{S}|c^\dagger_\mu(m)c_\tau(t)|\mathfrak{S}\rangle - \langle\mathfrak{S}|c^\dagger_\mu(m)c^\dagger_\sigma(s)c_\nu(n)c_\tau(t)|\mathfrak{S}\rangle, \tag{7.10}$$

where the two-point average $\langle c^\dagger_\mu(m)c_\tau(t)\rangle$ is expressible in terms of $\hat{D}^{\mathrm{cl}}_{\tau\mu}(\boldsymbol{k})$, while for calculating the four-point average we proceed as follows. Combine the spin and site indices into multi-index $M \equiv (\mu, m)$ (for details see the text after (5.19) and up to (5.21) included) and write the four-point average as

$$\langle\mathfrak{S}|c^\dagger_M c^\dagger_S c_N c_T|\mathfrak{S}\rangle = \langle\mathfrak{S}_0|\big[e^{-iW}c^\dagger_M e^{+iW}\big]\big[e^{-iW}c^\dagger_S e^{+iW}\big]\big[e^{-iW}c_N e^{+iW}\big]\big[e^{-iW}c_T e^{+iW}\big]|\mathfrak{S}_0\rangle =$$

$$= (U^\dagger)_{M'M}(U^\dagger)_{S'S}(U)_{NN'}(U)_{TT'}\langle\mathfrak{S}_0|c^\dagger_{M'}c^\dagger_{S'}c_{N'}c_{T'}|\mathfrak{S}_0\rangle. \tag{7.11}$$

Provided $|\mathfrak{S}_0\rangle$ is pure state, the four-point average in the right hand side is easily calculable and appears as

$$\langle\mathfrak{S}_0|c^\dagger_{M'}c^\dagger_{S'}c_{N'}c_{T'}|\mathfrak{S}_0\rangle = \nu_{N'}\nu_{T'}(\delta_{M'T'}\delta_{S'N'} - \delta_{M'N'}\delta_{S'T'}), \tag{7.12}$$

and we rewrite (7.10) as

$$\langle\mathfrak{S}|c^\dagger_M c^\dagger_S c_N c_T|\mathfrak{S}\rangle = \Big[\nu_{M'}(U)_{TM'}(U^\dagger)_{M'M}\Big]\Big[\nu_{S'}(U)_{NS'}(U^\dagger)_{S'S}\Big] -$$

$$-\Big[\nu_{M'}(U)_{NM'}(U^\dagger)_{M'M}\Big]\Big[\nu_{S'}(U)_{TS'}(U^\dagger)_{S'S}\Big]. \tag{7.13}$$

Comparing this to (5.25) we obtain

$$\langle\mathfrak{S}|c^\dagger_M c^\dagger_S c_N c_T|\mathfrak{S}\rangle = \langle\mathfrak{S}|c^\dagger_M c_T|\mathfrak{S}\rangle\langle\mathfrak{S}|c^\dagger_S c_N|\mathfrak{S}\rangle - \langle\mathfrak{S}|c^\dagger_M c_N|\mathfrak{S}\rangle\langle\mathfrak{S}|c^\dagger_S c_T|\mathfrak{S}\rangle. \tag{7.14}$$

Splitting multi-indices back into the spin and site ones we come to

$$\langle\mathfrak{S}|c^\dagger_\mu(m)c^\dagger_\sigma(s)c_\nu(n)c_\tau(t)|\mathfrak{S}\rangle = \langle\mathfrak{S}|c^\dagger_\mu(m)c_\tau(t)|\mathfrak{S}\rangle\langle\mathfrak{S}|c^\dagger_\nu(n)c_\sigma(s)|\mathfrak{S}\rangle -$$

$$-\langle\mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)|\mathfrak{S}\rangle\langle\mathfrak{S}|c^\dagger_\sigma(s)c_\tau(t)|\mathfrak{S}\rangle, \tag{7.15}$$

i.e. the four-point averages are expressed in terms of two-point ones. Remark, that this is an exact relation which is not the case for all Fock state. Here it has been derived for the particular class of states set by (5.15) and (5.16).

By use of (7.14) we write (7.9) as

$$\langle \mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)c^\dagger_\sigma(s)c_\tau(t)|\mathfrak{S}\rangle = \langle \mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)|\mathfrak{S}\rangle\langle \mathfrak{S}|c^\dagger_\sigma(s)c_\tau(t)|\mathfrak{S}\rangle - \\ - \langle \mathfrak{S}|c^\dagger_\mu(m)c_\tau(t)|\mathfrak{S}\rangle\langle \mathfrak{S}|c^\dagger_\nu(n)c_\sigma(s)|\mathfrak{S}\rangle + \\ + \delta_{\sigma\nu}\delta_{sn}\langle \mathfrak{S}|c^\dagger_\mu(m)c_\tau(t)|\mathfrak{S}\rangle. \tag{7.16}$$

7.3 Direct and Exchange Energies

Substituting (7.16) into (7.5) we obtain

$$\tfrac{1}{2}\int V(\boldsymbol{r}_1-\boldsymbol{r}_2)\langle \mathfrak{S}|\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)|\mathfrak{S}\rangle d\boldsymbol{r}_1 d\boldsymbol{r}_2 = \sum_{mn}\sum_{ij} V_{mnij}\langle \mathfrak{S}|c^\dagger_\mu(m)c_\mu(n)|\mathfrak{S}\rangle\cdot\langle \mathfrak{S}|c^\dagger_\nu(i)c_\nu(j)|\mathfrak{S}\rangle - \\ - \sum_{mn}\sum_{ij} V_{mnij}\langle \mathfrak{S}|c^\dagger_\mu(m)c_\nu(j)|\mathfrak{S}\rangle\cdot\langle \mathfrak{S}|c^\dagger_\nu(i)c_\mu(n)|\mathfrak{S}\rangle + \\ + \sum_{mn}\sum_{ij} V_{mnij}\delta_{ni}\langle \mathfrak{S}|c^\dagger_\mu(m)c_\mu(j)|\mathfrak{S}\rangle. \tag{7.17}$$

By use of (7.6) we modify the first term of this expression into

$$\sum_{mn}\sum_{ij} V_{mnij}\langle \mathfrak{S}|c^\dagger_\mu(m)c_\mu(n)|\mathfrak{S}\rangle\cdot\langle \mathfrak{S}|c^\dagger_\nu(i)c_\nu(j)|\mathfrak{S}\rangle = \tfrac{1}{2}\int V(\boldsymbol{r}_1-\boldsymbol{r}_2)\rho_{\rm cl}(\boldsymbol{r}_1)\rho_{\rm cl}(\boldsymbol{r}_2)d\boldsymbol{r}_1 d\boldsymbol{r}_2. \tag{7.18}$$

Collecting everything in (7.2) we come to

$$E = \underbrace{\tfrac{1}{2}\int V(\boldsymbol{r}_1-\boldsymbol{r}_2)\big[\rho_{\rm cl}(\boldsymbol{r}_1)-\rho_e\big]\big[\rho_{\rm cl}(\boldsymbol{r}_2)-\rho_e\big]d\boldsymbol{r}_1 d\boldsymbol{r}_2}_{\text{direct interaction energy } E_D} + \\ + \underbrace{\sum_{mn}\sum_{ij} V_{mnij}\delta_{ni}\langle \mathfrak{S}|c^\dagger_\mu(m)c_\mu(j)|\mathfrak{S}\rangle - \sum_{mn}\sum_{ij} V_{mnij}\langle \mathfrak{S}|c^\dagger_\mu(m)c_\nu(j)|\mathfrak{S}\rangle\cdot\langle \mathfrak{S}|c^\dagger_\nu(i)c_\mu(n)|\mathfrak{S}\rangle}_{\text{exchange interaction energy } E_X}, \tag{7.19}$$

i.e. the electron-electron interaction energy is split into direct and exchange energies.

Rewriting the direct energy in the Fourier form we obtain

$$E_D = \pi\int V(\boldsymbol{k})\big[\rho_{\rm cl}(-\boldsymbol{k})-2\pi\rho_e\delta(-\boldsymbol{k})\big]\big[\rho_{\rm cl}(\boldsymbol{k})-2\pi\rho_e\delta(\boldsymbol{k})\big]d\boldsymbol{k} = \\ = \pi\int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})\big[\hat\rho_{\rm cl}(-\boldsymbol{k})-2\pi\rho_e\delta(-\boldsymbol{k})\big]\big[\hat\rho_{\rm cl}(\boldsymbol{k})-2\pi\rho_e\delta(\boldsymbol{k})\big]d\boldsymbol{k}. \tag{7.20}$$

We proceed to express the exchange energy in terms of field averages. Substituting (7.8) we write

$$E_X = \frac{1}{4\pi}\int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})d\boldsymbol{k}\sum_m\langle \mathfrak{S}|c^\dagger_\mu(m)c_\mu(m)|\mathfrak{S}\rangle - \\ - \sum_{mn}\sum_{st}\frac{1}{4\pi}\int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})\langle m|e^{+i\boldsymbol{k}\boldsymbol{X}}|n\rangle\langle s|e^{-i\boldsymbol{k}\boldsymbol{X}}|t\rangle\langle \mathfrak{S}|c^\dagger_\mu(m)c_\nu(t)|\mathfrak{S}\rangle\langle \mathfrak{S}|c^\dagger_\nu(s)c_\mu(n)|\mathfrak{S}\rangle d\boldsymbol{k}, \tag{7.21}$$

where in the first line we used the fact that the states $|n\rangle$ form the complete set in the lowest Landau level.

We now need to express $\langle\mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)|\mathfrak{S}\rangle$ in terms of field averages. For this purpose we use

$$\int\langle m|e^{-i\boldsymbol{k}\boldsymbol{X}}|n\rangle\langle s|e^{+i\boldsymbol{k}\boldsymbol{X}}|t\rangle d\boldsymbol{k}=2\pi\ell^{-2}\delta_{mt}\delta_{ns},\tag{7.22}$$

which allows to revert (4.26) into

$$c^\dagger_\mu(m)c_\nu(n)=\ell^2\int\langle n|e^{+i\boldsymbol{k}\boldsymbol{X}}|m\rangle\hat{D}_{\nu\mu}(\boldsymbol{k})d\boldsymbol{k},\tag{7.23}$$

where from we obtain

$$\langle\mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)|\mathfrak{S}\rangle=\ell^2\int\langle n|e^{+i\boldsymbol{k}\boldsymbol{X}}|m\rangle\hat{D}^{\rm cl}_{\nu\mu}(\boldsymbol{k})d\boldsymbol{k}.\tag{7.24}$$

Adjusting this for using in the first line of (7.21) we obtain

$$\begin{aligned}\sum_n\langle\mathfrak{S}|c^\dagger_\mu(n)c_\mu(n)|\mathfrak{S}\rangle&=2\pi\hat{D}^{\rm cl}_{\mu\mu}(\boldsymbol{k}\to 0)=\\&=2\pi\ell^2\int\hat{D}^{\rm cl}_{\mu\sigma}(-\boldsymbol{k}')\hat{D}^{\rm cl}_{\sigma\mu}(\boldsymbol{k}')d\boldsymbol{k}'\end{aligned}\tag{7.25}$$

where the second equality follows from (5.30).

We then rewrite (7.19) as

$$\begin{aligned}E_X=\frac{\ell^2}{2}\int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})d\boldsymbol{k}\int\hat{D}^{\rm cl}_{\mu\sigma}(-\boldsymbol{k}')\hat{D}^{\rm cl}_{\sigma\mu}(\boldsymbol{k}')d\boldsymbol{k}'-\\-\frac{\ell^4}{4\pi}\sum_n\int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})\langle n|e^{+i\boldsymbol{k}''\boldsymbol{X}}e^{-i\boldsymbol{k}\boldsymbol{X}}e^{+i\boldsymbol{k}'\boldsymbol{X}}e^{+i\boldsymbol{k}\boldsymbol{X}}|n\rangle\hat{D}^{\rm cl}_{\nu\mu}(\boldsymbol{k}')\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{k}'')d\boldsymbol{k}d\boldsymbol{k}'d\boldsymbol{k}''.\end{aligned}\tag{7.26}$$

Employing the Baker-Campbell-Hausdorff formula together with (3.22) we find

$$\sum_n\langle n|e^{+i\boldsymbol{k}''\boldsymbol{X}}e^{-i\boldsymbol{k}\boldsymbol{X}}e^{+i\boldsymbol{k}'\boldsymbol{X}}e^{+i\boldsymbol{k}\boldsymbol{X}}|n\rangle=2\pi\ell^{-2}e^{-i\ell^2\boldsymbol{k}\wedge\boldsymbol{k}'}\delta(\boldsymbol{k}'+\boldsymbol{k}''),\tag{7.27}$$

and pass to

$$E_X=\tfrac{1}{2}\pi\int\big[V_X(0)-V_X(\boldsymbol{k})\big]\hat{D}^{\rm cl}_{\mu\nu}(-\boldsymbol{k})\hat{D}^{\rm cl}_{\nu\mu}(\boldsymbol{k})d\boldsymbol{k},\tag{7.28}$$

where we have introduced the notation

$$V_X(\boldsymbol{k})=\frac{\ell^2}{\pi}\int e^{+i\ell^2\boldsymbol{k}\wedge\boldsymbol{k}'}e^{-\frac{1}{2}\ell^2\boldsymbol{k}'^2}V(\boldsymbol{k}')d\boldsymbol{k}'.\tag{7.29}$$

As follows from (5.32) we have

$$\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{k})=\frac{1}{N}\delta_{\mu\nu}\hat{\rho}_{\rm cl}(\boldsymbol{k})+(\lambda_a)_{\mu\nu}\hat{I}^{\rm cl}_a(\boldsymbol{k}).\tag{7.30}$$

Combining this with (4.15) we obtain

$$\hat{D}^{\rm cl}_{\mu\nu}(-\boldsymbol{k})\hat{D}^{\rm cl}_{\nu\mu}(\boldsymbol{k})=\frac{1}{N}\hat{\rho}_{\rm cl}(-\boldsymbol{k})\hat{\rho}_{\rm cl}(\boldsymbol{k})+2\hat{I}^{\rm cl}_a(-\boldsymbol{k})\hat{I}^{\rm cl}_a(\boldsymbol{k}),\tag{7.31}$$

and the final expression for the exchange energy appears as

$$E_X=\pi\int\big[V_X(0)-V_X(\boldsymbol{k})\big]\Big\{\hat{I}^{\rm cl}_a(-\boldsymbol{k})\hat{I}^{\rm cl}_a(\boldsymbol{k})+\frac{1}{2N}\hat{\rho}_{\rm cl}(-\boldsymbol{k})\hat{\rho}_{\rm cl}(\boldsymbol{k})\Big\}d\boldsymbol{k}.\tag{7.32}$$

For the hard-core and Coulomb interactions we have

$$V(\boldsymbol{r}_1-\boldsymbol{r}_2)=\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\qquad\Rightarrow\qquad V(\boldsymbol{k})=\frac{1}{2\pi}\qquad\Rightarrow\qquad V_X(\boldsymbol{k})=\frac{1}{\pi}e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2},\tag{7.33a}$$

$$V(\boldsymbol{r}_1-\boldsymbol{r}_2)=\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\qquad\Rightarrow\qquad V(\boldsymbol{k})=\frac{1}{k}\qquad\Rightarrow\qquad V_X(\boldsymbol{k})=\sqrt{2\pi}\ell e^{-\frac{1}{4}\ell^2\boldsymbol{k}^2}I_0(\tfrac{1}{4}\ell^2\boldsymbol{k}^2),\tag{7.33b}$$

where I_0 is the modified Bessel function.

7.4 Exchange Energy and Non-Commutative CP Models

We take closer look at the exchange energy. In particular, we proceed to reveal the non-commutative constructions hidden in the exchange energy.

For this purpose we consider the system at $\nu = 1$ described by the Fock state $|\mathfrak{S}\rangle$. The corresponding core field averages $\hat{\rho}_{\rm cl}(\boldsymbol{r}) = \mathfrak{S}|\hat{\rho}(\boldsymbol{r})|\mathfrak{S}\rangle$ and $\hat{I}^{\rm cl}_a(\boldsymbol{r}) = \langle\mathfrak{S}|\hat{I}_a(\boldsymbol{r})|\mathfrak{S}\rangle$ can be expressed via the underlying non-commutative CP^{N-1} field $f_\mu(\boldsymbol{r})$. Let $\mathfrak{f}_\mu \equiv O[f_\mu]$ be the corresponding Weyl operator. We then have the relation

$$\langle\mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)|\mathfrak{S}\rangle = \langle n|\mathfrak{f}_\nu\mathfrak{f}^\dagger_\mu|m\rangle. \tag{7.34}$$

Rigorous derivation of this relation is bit lengthy, and therefore here we only verify its validity. In particular, we know that the quantities $\langle\mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)|\mathfrak{S}\rangle$ must satisfy the relation (5.29). We demonstrate that (5.29) follows from (6.34) what we accept as verification of the later. We have

$$\sum_{\sigma=1}^{N}\sum_{s=0}^{\infty}\langle\mathfrak{S}|c^\dagger_\mu(m)c_\sigma(s)|\mathfrak{S}\rangle\langle\mathfrak{S}|c^\dagger_\sigma(s)c_\nu(n)|\mathfrak{S}\rangle = \sum_{\sigma=1}^{N}\sum_{s=0}^{\infty}\langle s|\mathfrak{f}_\sigma\mathfrak{f}^\dagger_\mu|m\rangle\langle n|\mathfrak{f}_\nu\mathfrak{f}^\dagger_\sigma|s\rangle =$$

$$= \sum_{\sigma=1}^{N}\sum_{s=0}^{\infty}\langle n|\mathfrak{f}_\nu\mathfrak{f}^\dagger_\sigma|s\rangle\langle s|\mathfrak{f}_\sigma\mathfrak{f}^\dagger_\mu|m\rangle = \sum_{\sigma=1}^{N}\sum_{s=0}^{\infty}\langle n|\mathfrak{f}_\nu\mathfrak{f}^\dagger_\sigma\mathfrak{f}_\sigma\mathfrak{f}^\dagger_\mu|m\rangle. \tag{7.35}$$

Using here the normalization condition $\mathfrak{f}^\dagger\mathfrak{f} = \mathbb{I}$ we come to

$$\sum_{\sigma=1}^{N}\sum_{s=0}^{\infty}\langle\mathfrak{S}|c^\dagger_\mu(m)c_\sigma(s)|\mathfrak{S}\rangle\langle\mathfrak{S}|c^\dagger_\sigma(s)c_\nu(n)|\mathfrak{S}\rangle = \langle n|\mathfrak{f}_\nu\mathfrak{f}^\dagger_\mu|m\rangle =$$

$$= \langle\mathfrak{S}|c^\dagger_\mu(m)c_\nu(n)|\mathfrak{S}\rangle. \tag{7.36}$$

Employing (6.34) together with (7.8) in (7.19) we find

$$E_X = \frac{1}{4\pi}\int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})\mathtt{Tr}\Big\{\mathfrak{f}_\mu\mathfrak{f}^\dagger_\mu - e^{+i\boldsymbol{k}\boldsymbol{X}}\mathfrak{f}_\mu\mathfrak{f}^\dagger_\nu e^{-i\boldsymbol{k}\boldsymbol{X}}\mathfrak{f}_\nu\mathfrak{f}^\dagger_\mu\Big\}d\boldsymbol{k}. \tag{7.37}$$

Remark that the two terms in braces cancel out each other for $\boldsymbol{k} = 0$, what in $\boldsymbol{r}$ space means the convergence of the corresponding integral. Therefore we may change the order of operators under the sign of trace so to maintain this cancelation. We then bring this expression to

$$E_X = \frac{1}{4\pi}\int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})\mathtt{Tr}\Big\{\mathbb{I} - \mathfrak{f}^\dagger_\mu e^{+i\boldsymbol{k}\boldsymbol{X}}\mathfrak{f}_\mu\mathfrak{f}^\dagger_\nu e^{-i\boldsymbol{k}\boldsymbol{X}}\mathfrak{f}_\nu\Big\}d\boldsymbol{k}. \tag{7.38}$$

Reformulating in terms of $f_\mu(\boldsymbol{r})$ this appears as

$$E_X = \frac{1}{8\pi^2\ell^2}\int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})d\boldsymbol{k}\int\Big\{1 - f^\dagger_\mu\star e^{+i\boldsymbol{k}\boldsymbol{r}}\star f_\mu\star f^\dagger_\nu\star e^{-i\boldsymbol{k}\boldsymbol{r}}\star f_\nu\Big\}d\boldsymbol{r}, \tag{7.39}$$

where the non-commutative structure is evident.

It is instructive to study the low-momentum expansion ($\boldsymbol{k}\sim 0$) in (7.38). This is done by expanding the exponents

$$e^{+i\boldsymbol{k}\boldsymbol{X}} = 1 + ik_nX_n - \tfrac{1}{2}k_mk_nX_mX_n + \mathcal{O}(k^3). \tag{7.40}$$

Substituting into (7.38) the terms linear in k_n disappear. Assuming the rotation invariance of $V(\boldsymbol{k})$ and integrating over the polar angle we come to

$$E_X = \underbrace{\tfrac{1}{4}\int k^3e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2}V(\boldsymbol{k})dk}_{\equiv J_s}\mathtt{Tr}\Big\{\mathfrak{f}^\dagger_\mu X_nX_n\mathfrak{f}_\mu - \mathfrak{f}^\dagger_\mu X_n\mathfrak{f}_\mu\mathfrak{f}^\dagger_\nu X_n\mathfrak{f}_\nu\Big\}. \tag{7.41}$$

By explicit calculation we can verify that this can be rewritten as

$$E_X = J_s\mathtt{Tr}\Big\{[\mathfrak{f}^\dagger_\mu, X_n][X_n,\mathfrak{f}_\mu] - \mathfrak{f}^\dagger_\mu[X_n,\mathfrak{f}_\mu]\mathfrak{f}^\dagger_\nu[X_n,\mathfrak{f}_\nu]\Big\}. \tag{7.42}$$

The reason for such formulation is that the commuting any Weyl operator $O[f]$ with X_n is equivalent to differentiating the corresponding symbol $f(\boldsymbol{r})$, see (3.31). We then rewrite (7.42) further as

$$E_X = J_s \ell^2 \mathtt{Tr} \Big\{ (\partial_n \mathfrak{f}_\mu^\dagger)(\partial_n \mathfrak{f}_\mu) + (\mathfrak{f}_\mu^\dagger \partial \mathfrak{f}_\mu)(\mathfrak{f}_\nu^\dagger \partial \mathfrak{f}_\nu) \Big\}. \tag{7.43}$$

Remark that the model defined by the energy functional (7.43) is known as the non-commutative CP model. This model used to be written artificially as a mere generalizations of the commutative CP models, while here we have shown that they emerge as the lowest-order term in the derivative expansion of the exchange four-fermion interaction, irrespectively of the particular form of $V(\boldsymbol{r})$.

We can use the non-commutative CP model (7.43) as an effective Hamiltonian for the Goldstone modes describing fluctuations of CP field around the ground state. Such fluctuations occur without density excitation $\rho^{\mathtt{cl}}(\boldsymbol{r}) - \rho_e = 0$, thus minimizing the direct energy as $E_D = 0$. Once the density fluctuations are absent, the CP field satisfies

$$\mathfrak{f}_\sigma \mathfrak{f}_\sigma^\dagger = \mathfrak{f}_\sigma^\dagger \mathfrak{f}_\sigma = \mathbb{I}, \tag{7.44}$$

as it appears in the definition of non-commutative CP models.

Problems

(7.a) Derive (7.8) from (7.6).

(7.b) Check (7.11).

(7.c) Derive (7.20).

(7.d) Derive (7.25).

Chapter 8

GROUND STATE AND ELEMENTARY EXCITATIONS

8.1 Ground State and Spontaneous Symmetry Breaking

We have shown that the energy of electron-electron interaction splits into direct and exchange pieces

$$E = E_D + E_X, \tag{8.1}$$

where E_D and E_X are given by (7.18) and (7.30) respectively. For our purposes it is reasonable to rewrite those in a bit different way. Since $\rho_{\rm cl}(\boldsymbol{r})$ is real we have $\rho_{\rm cl}(-\boldsymbol{k}) = [\rho_{\rm cl}(\boldsymbol{k})]^*$ *i.e.* $\hat{\rho}_{\rm cl}(-\boldsymbol{k}) = [\hat{\rho}_{\rm cl}(\boldsymbol{k})]^*$. The same is true for isospin fields, *i.e.* $\hat{I}^{\rm cl}_a(-\boldsymbol{k}) = [\hat{I}^{\rm cl}_a(\boldsymbol{k})]^*$. Then the direct and exchange energies can be written as

$$E_D = \pi \int e^{-\frac{1}{2}\ell^2\boldsymbol{k}^2} V(\boldsymbol{k}) \big|\hat{\rho}_{\rm cl}(\boldsymbol{k}) - 2\pi\rho_e \delta(\boldsymbol{k})\big|^2 d\boldsymbol{k}, \tag{8.2a}$$

$$E_X = \pi \int \big[V_X(0) - V_X(\boldsymbol{k})\big] \left\{ \big|\hat{I}^{\rm cl}_a(\boldsymbol{k})\big|^2 + \frac{1}{2N}\big|\hat{\rho}_{\rm cl}(\boldsymbol{k})\big|^2 \right\} d\boldsymbol{k}. \tag{8.2b}$$

It is remarkable that, though the Hamiltonian (7.1) does not involve the isospin variables, the energy of a Fock state does so, and this dependence arises due to exchange interaction.

From these expressions we can already carry out physical conclusions about the structure of the ground state. Let us first point out that both of (8.2) are non-negatively defined if the interaction is repulsive. Indeed, for repulsive interactions we have $V(\boldsymbol{k}) \geqslant 0$, hence the direct energy is explicitly non-negative. For the exchange energy potential $V_X(\boldsymbol{k})$ we have

$$\big|V_X(\boldsymbol{k})\big| = \frac{\ell^2}{\pi} \left| \int e^{+i\ell^2 \boldsymbol{k}\wedge\boldsymbol{k}'} e^{-\frac{1}{2}\ell^2\boldsymbol{k}'^2} V(\boldsymbol{k}') d\boldsymbol{k}' \right| \leqslant$$

$$\leqslant \frac{\ell^2}{\pi} \int \left| e^{+i\ell^2\boldsymbol{k}\wedge\boldsymbol{k}'} V(\boldsymbol{k}')\, e^{-\frac{1}{2}\ell^2\boldsymbol{k}'^2} \right| d\boldsymbol{k}' = \frac{\ell^2}{\pi} \int V(\boldsymbol{k}')\, e^{-\frac{1}{2}\ell^2\boldsymbol{k}'^2}\, d\boldsymbol{k}' = V_X(0). \tag{8.3}$$

Provided $V_X(0) > 0$ we have $V_X(0) - V_X(\boldsymbol{k}) > 0$, hence the exchange energy is also non-negative.

From $E_D \geqslant 0$ it follows that the lower bound of the direct energy is zero, and this bound is attained at

$$\hat{\rho}_{\rm cl}(\boldsymbol{k}) = 2\pi\rho_L \nu \delta(\boldsymbol{k}), \tag{8.4}$$

what implies the density is homogeneous $\rho_{\rm cl}(\boldsymbol{r}) = \rho_e = \rho_L \nu$. We consider the cases of integer fillings what corresponds to ν being an integer. Provided an electron can have N different polarizations ($\mu = 1, 2, \ldots, N$) the maximal value of ν is N. Fillings with $\nu = 0$ and $\nu = N$ are trivial: $\nu = 0$ implies there are no electrons at all, while $\nu = N$ means that no vacant states are available what excludes any nontrivial dynamics because of Pauli exclusion principle. Therefore we assume $\nu = 1, 2, \ldots, N-1$.

The relation (8.4) implies that every Landau site accommodates exactly ν electrons, but says nothing about the polarizations of those electrons. The later is governed by E_X to be minimal. This is attained at

$$\hat{I}^{\rm cl}_a(\boldsymbol{k}) = 2\pi\rho_L \mathcal{J}_a \delta(\boldsymbol{k}), \tag{8.5}$$

where $\mathcal{J}_a$ is an arbitrary constant vector. Indeed, the relation (8.5) together with (8.4) leads to $E_X = 0$ what is the lower bound of the exchange energy.

Summarizing, the configurations (8.4) and (8.5) correspond to the absolute minimum of $E = \langle \mathfrak{S}|H_e|\mathfrak{S}\rangle$ what is the true ground state of the system. Our results indicate that the ground state is attained when all isospins are aligned

along an arbitrary direction $\mathcal{J}_a$ in $SU(N)$ space. This is the manifestation of spontaneous symmetry breaking which in this particular system occurs due to the exchange interaction hidden in the four-fermion interaction (7.1).

We now present the Fock state corresponding to the ground state set by (8.4) and (8.5). For the sake of simplicity we take $\nu = 1$. In this case each Landau site comprises one electron and the Fock state looks as

$$|\mathrm{g}\rangle = \prod_{n=0}^{\infty} g_\mu c_\mu^\dagger(n) \cdot |\varnothing\rangle, \tag{8.6}$$

where the n-independent vector g_μ determines the polarization in isospin space. It is normalized as

$$\bar{g}_\sigma g_\sigma = 1, \tag{8.7}$$

so the ground state (8.6) is normalized as $\langle \mathrm{g}|\mathrm{g}\rangle = 1$.

The state (8.6) is the eigenstate of the particle density operator. If more precisely, we have the relation

$$c_\mu^\dagger(m) c_\mu(n)|\mathrm{g}\rangle = \delta_{mn}|\mathrm{g}\rangle, \tag{8.8}$$

what leads to

$$\rho(\boldsymbol{r})|\mathrm{g}\rangle = \rho_L|\mathrm{g}\rangle, \tag{8.9}$$

and further to

$$H_e|\mathrm{g}\rangle = 0. \tag{8.10}$$

The later implies that (8.6) is the eigenstate of the interaction Hamiltonian (7.1).

Further we have

$$\langle \mathrm{g}|c_\mu^\dagger(m) c_\nu(n)|\mathrm{g}\rangle = \bar{g}_\mu g_\nu \delta_{mn}, \tag{8.11}$$

where from we come to (8.4) with $\nu = 1$ and to (8.5) with $\mathcal{J}_a = \frac{1}{2} g^\dagger \lambda_a g$. In the configuration space these lead to

$$\rho_{\mathrm{cl}}(\boldsymbol{r}) = \hat{\rho}_{\mathrm{cl}}(\boldsymbol{r}) = \rho_L, \tag{8.12a}$$

$$I_a^{\mathrm{cl}}(\boldsymbol{r}) = \hat{I}_a^{\mathrm{cl}}(\boldsymbol{r}) = \tfrac{1}{2}\rho_L g^\dagger \lambda_a g. \tag{8.12b}$$

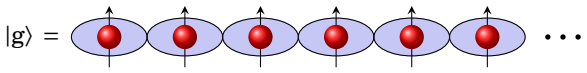
Remark, that the core and physical fields coincide since they are all spatially homogeneous.

8.2 Hole and Electron Excitations

In previous chapter we have introduced the Fock state corresponding to the ground state of N-component electron state at filling $\nu = 1$. We now proceed to discuss the Fock states representing elementary excitation emerging over this ground state. First, we consider the case of $N = 2$, *i.e.* up-spin and down-spin electrons, hence the only nontrivial filling is $\nu = 1$. Second, we assume the vector g_μ determining the polarization of the ground state is $(g_\uparrow, g_\downarrow) = (1, 0)$, *i.e.* spins are polarized in z-direction (even if it is not so, we can perform the rotation of spin space thus aligning g_μ along the spin z-axis). Thus the ground state appears as

$$|\mathrm{g}\rangle = \prod_{n=0}^{\infty} c_\uparrow^\dagger(n) \cdot |\varnothing\rangle, \tag{8.13}$$

what can be presented graphically as



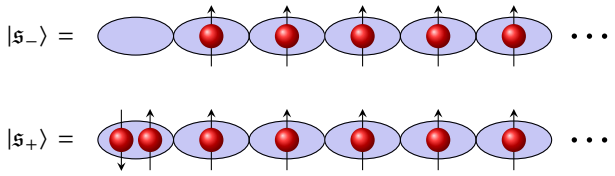
In accord with (8.12) the corresponding only nonvanishing classical fields are

$$\rho_{\mathrm{cl}}(\boldsymbol{r}) = \hat{\rho}_{\mathrm{cl}}(\boldsymbol{r}) = \rho_L, \tag{8.14a}$$

$$I_z^{\mathrm{cl}}(\boldsymbol{r}) = \hat{I}_z^{\mathrm{cl}}(\boldsymbol{r}) = \tfrac{1}{2}\rho_L. \tag{8.14b}$$

We first discuss the simplest excited stats which are hole and electron states and the generalize these to skyrmion and antiskyrmion states.

Hole-excited state is obtained by removing an electron from the ground state. For the sake of simplicity we assume this is the electron at $n = 0$. Electron state is constructed by accommodating an extra electron on the site with $n = 0$. Provided $n = 0$ is already occupied by an up-spin electron only down-spin electron can be added. These can be depicted graphically as



These are expressed analytically by

$$|\mathfrak{s}_-\rangle = c_\uparrow(0)|\mathrm{g}\rangle = \prod_{n=1}^{\infty} c_\uparrow^\dagger(n)\cdot|\varnothing\rangle, \tag{8.15a}$$

$$|\mathfrak{s}_+\rangle = c_\downarrow^\dagger(0)|\mathrm{g}\rangle = c_\downarrow^\dagger(0)\prod_{n=0}^{\infty} c_\uparrow^\dagger(n)\cdot|\varnothing\rangle, \tag{8.15b}$$

which obviously belong to the class of Fock states set by (5.15) and (5.16). Therefore the corresponding direct and exchange energies can be calculated by use of (8.2), hence we only need to calculate the averages $\hat\rho_{\mathrm{cl}}(\boldsymbol{k})$ and $\hat I_\alpha^{\mathrm{cl}}(\boldsymbol{k})$.

For this purpose we use

$$\langle\mathfrak{s}_-|c_\mu^\dagger(m)c_\nu(n)|\mathfrak{s}_-\rangle = \delta_{\mu\uparrow}\delta_{\nu\uparrow}(\delta_{mn}-\delta_{m0}\delta_{n0}), \tag{8.16a}$$

$$\langle\mathfrak{s}_+|c_\mu^\dagger(m)c_\nu(n)|\mathfrak{s}_+\rangle = \delta_{\mu\uparrow}\delta_{\nu\uparrow}\delta_{mn}+\delta_{\mu\downarrow}\delta_{\nu\downarrow}\delta_{m0}\delta_{n0}. \tag{8.16b}$$

We then use

$$\hat D_{\mu\nu}^{\mathrm{cl}}(\boldsymbol{k}) = \frac{1}{2\pi}\sum_{mn}\langle m|e^{-i\boldsymbol{k}\boldsymbol{X}}|n\rangle\langle\mathfrak{s}_\pm|c_\nu^\dagger(m)c_\mu(n)|\mathfrak{s}_\pm\rangle \tag{8.17}$$

and find

$$\hat D_{\mu\nu}^{\mathrm{cl}-}(\boldsymbol{k}) = \delta_{\mu\uparrow}\delta_{\nu\uparrow}\ell^{-2}\delta(\boldsymbol{k})-\frac{1}{2\pi}\delta_{\mu\uparrow}\delta_{\nu\uparrow}e^{-\frac14\ell^2\boldsymbol{k}^2}, \tag{8.18a}$$

$$\hat D_{\mu\nu}^{\mathrm{cl}+}(\boldsymbol{k}) = \delta_{\mu\uparrow}\delta_{\nu\uparrow}\ell^{-2}\delta(\boldsymbol{k})+\frac{1}{2\pi}\delta_{\mu\downarrow}\delta_{\nu\downarrow}e^{-\frac14\ell^2\boldsymbol{k}^2}, \tag{8.18b}$$

where we have used $\langle n|e^{i\boldsymbol{k}\boldsymbol{X}}|n\rangle = e^{-\frac14\ell^2\boldsymbol{k}^2}L_n(\frac12\ell^2\boldsymbol{k}^2)$ and $\mathtt{Tr}\big[e^{i\boldsymbol{k}\boldsymbol{X}}\big] = 2\pi\ell^{-2}\delta(\boldsymbol{k})$; see (3.17) and (3.22).

From (8.5) we find $\hat I_x^{\mathrm{cl}}(\boldsymbol{k}) = \hat I_y^{\mathrm{cl}}(\boldsymbol{k}) = 0$ while the only nonvanishing averages are

$$\hat\rho_{\mathrm{cl}}^{\pm}(\boldsymbol{k}) = \frac{1}{\ell^2}\delta(\boldsymbol{k})\pm\frac{1}{2\pi}e^{-\frac14\ell^2\boldsymbol{k}^2}, \tag{8.19a}$$

$$\hat I_z^{\mathrm{cl}\pm}(\boldsymbol{k}) = \frac{1}{2\ell^2}\delta(\boldsymbol{k})-\frac{1}{4\pi}e^{-\frac14\ell^2\boldsymbol{k}^2}, \tag{8.19b}$$

where the first terms are the ground state values, hence the second ones represent the excitation parts.

In the configuration space these appear as

$$\hat\rho_{\mathrm{cl}}^{\pm}(\boldsymbol{r}) = \frac{1\pm2e^{-r^2/\ell^2}}{2\pi\ell^2} \qquad\Rightarrow\qquad \rho_{\mathrm{cl}}^{\pm}(\boldsymbol{r}) = \frac{1\pm e^{-\frac12 r^2/\ell^2}}{2\pi\ell^2}, \tag{8.20a}$$

$$\hat I_z^{\mathrm{cl}\pm}(\boldsymbol{r}) = \frac{1-2e^{-r^2/\ell^2}}{4\pi\ell^2} \qquad\Rightarrow\qquad I_z^{\mathrm{cl}\pm}(\boldsymbol{r}) = \frac{1-e^{-\frac12 r^2/\ell^2}}{4\pi\ell^2}. \tag{8.20b}$$

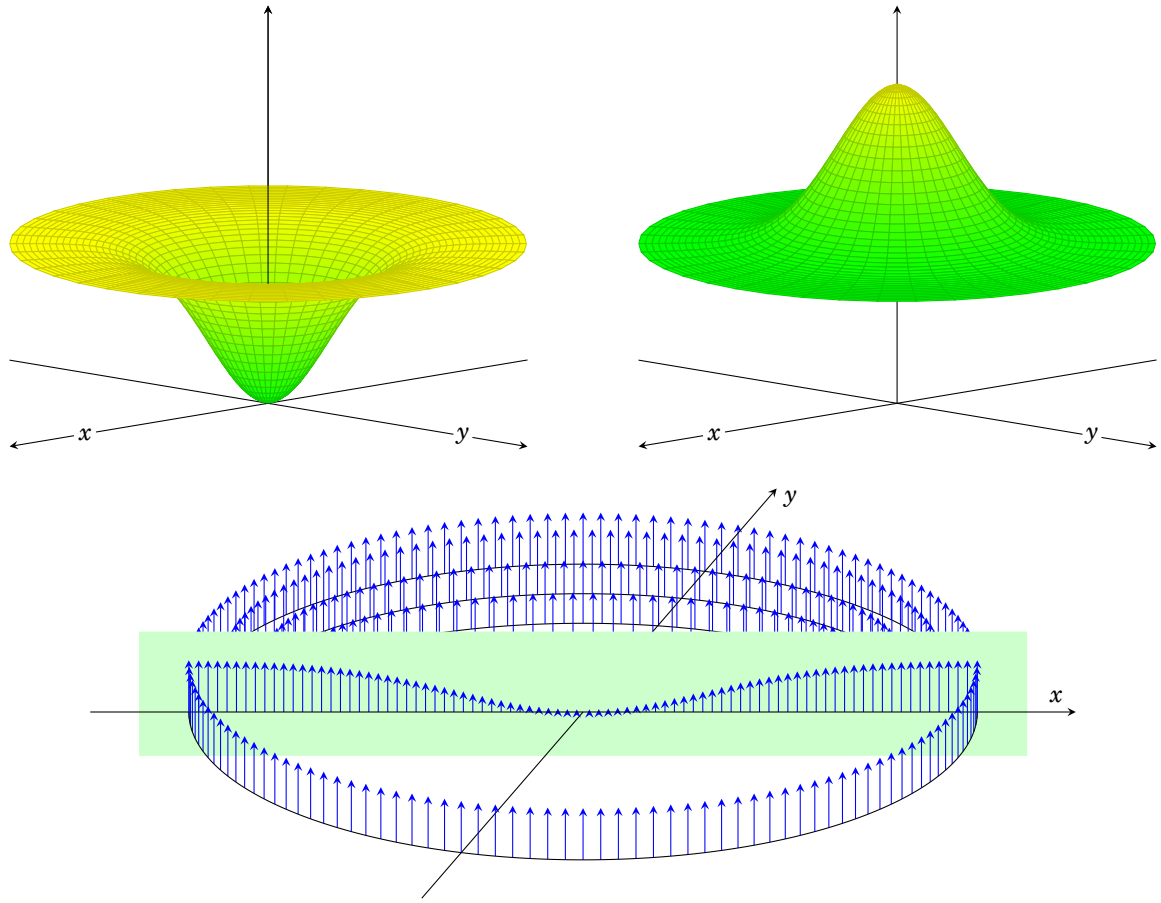


Fig. 8.1: Upper panel shows particle densities for hole (left) and electron (right) excitations. Lower panel shows spin distribution which is identical for hole and electron excitations.

Remark, that the core density of hole-excited state may become negative, while the corresponding physical one is strictly positive. In figure (8.1) we present the spatial plots for density and spin textures.

Total interaction energies ($E = E_D + E_X$) for hole and electron states are identical and look as

$$E^{\pm} = \frac{1}{4\pi} \int e^{-\frac{1}{2}\ell^2 \boldsymbol{k}^2} V(\boldsymbol{k}) d\boldsymbol{k}. \tag{8.21}$$

Thus, removing an electron from the ground state or adding to it causes the increase of Coulomb energy. Such an excitation will expectedly affect neighbouring electrons, which may modify their spin polarizations coherently and thus lower the Coulomb energy. The resulting configurations are referred to as the quantum Hall skyrmions discussed in the next section.

8.3 Skyrmions

Originally the Fock state describing coherent excitations of spins in the lowest Landau level was proposed by Fertig *et al.* (*PRB* **50**, 1994) in the form

$$|\mathfrak{S}_-\rangle = \left\{ \prod_{n=0}^{\infty} \left[c_{\downarrow}^{\dagger}(n) \sin\vartheta_n + c_{\uparrow}^{\dagger}(n+1) \cos\vartheta_n \right] \right\} |\varnothing\rangle, \tag{8.22}$$

where ϑ_n are real parameters to be fixed by minimizing the Coulomb energy.

Remind that larger is n, the corresponding one-particle state is localized more away from the origin. Therefore, requiring the core field averages with respect to (8.22) approach the ground state values at spatial infinity we come to the condition

$$\lim_{n\to\infty} \vartheta_n = 0. \tag{8.23}$$

In that case the large-n content of (8.22) consists of exclusively up-spin electrons and consequently reproduces the ground state at spatial infinity.

In what follows we show that the skyrmion state (8.22) can be brought to the form given by (5.15) and (5.16). Motivation is twofold: (a) once the skyrmion Fock states fall in that category, all the results obtained for that category can be employed; (b) this will reveal the algebraic structure of quantum Hall skyrmions what makes the essence of these configurations more transparent.

Consider the element of W-algebra given by

$$W^- = \sum_{n=0}^{\infty} W^-(n), \tag{8.24}$$

where

$$iW^-(n) = \vartheta_n \big[c_\downarrow^\dagger(n) c_\uparrow(n+1) - c_\uparrow^\dagger(n+1) c_\downarrow(n) \big], \tag{8.25}$$

with ϑ_n the ones introduced in (8.22).

The operator e^{iW^-} can be factorized

$$e^{iW^-} = e^{iW^-(0)+iW^-(1)+iW^-(2)+\cdots} = e^{iW^-(0)} e^{iW^-(1)} e^{iW^-(2)} \cdots, \tag{8.26}$$

due to the fact that $\big[W^-(m), W^-(n)\big] = 0$.

Apply the operator e^{iW^-} to the hole-excited state (8.15a). Employing $e^{iW^-} e^{-iW^-} = 1$ together with $e^{iW^-}|\varnothing\rangle = |\varnothing\rangle$ we proceed as follows

$$\begin{aligned} e^{iW^-}|\mathfrak{s}_-\rangle &= e^{iW^-}\big[c_\uparrow^\dagger(1) c_\uparrow^\dagger(2) c_\uparrow^\dagger(3)\cdots\big]|\varnothing\rangle = \\ &= \Big\{\big[e^{iW^-} c_\uparrow^\dagger(1) e^{-iW^-}\big]\big[e^{iW^-} c_\uparrow^\dagger(2) e^{-iW^-}\big]\big[e^{iW^-} c_\uparrow^\dagger(3) e^{-iW^-}\big]\Big\}|\varnothing\rangle = \prod_{n=0}^{\infty} \xi^\dagger(n)\cdot|\varnothing\rangle, \end{aligned} \tag{8.27}$$

where

$$\xi^\dagger(n) \equiv e^{iW^-} c_\uparrow^\dagger(n+1) e^{-iW^-}. \tag{8.28}$$

Remark, that $[W^-(m), c_\uparrow^\dagger(n+1)] = 0$ for all m except when $m = n$. Besides due to $\big[W^-(m), W^-(n)\big] = 0$ all multiplicatives in the right hand side of (8.16) commute among each other and therefore can be arranged arbitrarily. We then have

$$\xi^\dagger(n) = e^{iW^-} c_\uparrow^\dagger(n+1) e^{-iW^-} = e^{iW^-(n)} c_\uparrow^\dagger(n+1) e^{-iW^-(n)}. \tag{8.29}$$

In order to calculate the right hand side of this expression let us differentiate with respect to ϑ_n

$$\begin{aligned} \dot\xi^\dagger(n) &= \frac{de^{iW^-(n)}}{d\vartheta_n} c_\uparrow^\dagger(n+1) e^{-iW^-(n)} + e^{iW^-(n)} c_\uparrow^\dagger(n+1) \frac{de^{-iW^-(n)}}{d\vartheta_n} = \\ &= e^{iW^-(n)} i\frac{dW^-(n)}{d\vartheta_n} c_\uparrow^\dagger(n+1) e^{-iW^-(n)} - e^{iW^-(n)} c_\uparrow^\dagger(n+1) i\frac{dW^-(n)}{d\vartheta_n} e^{-iW^-(n)} = \\ &= e^{iW^-(n)} \left[i\frac{dW^-(n)}{d\vartheta_n}, c_\uparrow^\dagger(n+1) \right] e^{-iW^-(n)} = \\ &= e^{iW^-(n)} \big[c_\downarrow^\dagger(n) c_\uparrow(n+1) - c_\uparrow^\dagger(n+1) c_\downarrow(n), c_\uparrow^\dagger(n+1) \big] e^{-iW^-(n)} = e^{iW^-(n)} c_\downarrow^\dagger(n) e^{-iW^-(n)}. \end{aligned} \tag{8.30}$$

Differentiating once again and performing similar manipulations we find

$$\ddot\xi^\dagger(n) = -e^{iW^-(n)} c_\uparrow^\dagger(n+1) e^{-iW^-(n)} \qquad \Rightarrow \qquad \ddot\xi^\dagger(n) + \xi^\dagger(n) = 0. \tag{8.31}$$

Form (8.28) and (8.30) we find that the values of $\xi^\dagger(n)$ and $\dot\xi^\dagger(n)$ at $\vartheta_n = 0$ are $c_\uparrow^\dagger(n+1)$ and $c_\downarrow^\dagger(n)$ respectively. Hence the differential equation (8.31) is integrated as

$$\xi^\dagger(n) = c_\downarrow^\dagger(n) \sin\vartheta_n + c_\uparrow^\dagger(n+1) \cos\vartheta_n, \tag{8.32}$$

i.e. the right hand side of (8.27) coincides with (8.22).

Summarizing, the skyrmion Fock state (8.22) is expressed as

$$|\mathfrak{S}_-\rangle = e^{iW^-}|\mathfrak{s}_-\rangle, \tag{8.33}$$

i.e the quantum Hall skyrmion is nothing else but the W-transformation of the hole-excited state, and the structure set by (5.15) and (5.16) is evident.

This construction can be generalized to multi-skyrmion (or q-skyrmion) states by removing q electrons from the ground state and then applying the W-transformation

$$|\mathfrak{S}_-^q\rangle = e^{iW^-}c_\uparrow(0)c_\uparrow(1)\cdots c_\uparrow(q-1)|\mathrm{g}\rangle, \tag{8.34a}$$

$$iW^- = \sum_n \vartheta_n\big[c_\downarrow^\dagger(n)c_\uparrow(n+q) - c_\uparrow^\dagger(n+q)c_\downarrow(n)\big]. \tag{8.34b}$$

Performing the manipulations analogous to (8.26) – (8.31) we can bring the q-skyrmion state (8.34a) to the form

$$|\mathfrak{S}_-^q\rangle = \prod_{n=0}^{\infty}\xi^\dagger(n)\cdot|\varnothing\rangle, \tag{8.35a}$$

$$\xi^\dagger(n) = c_\downarrow^\dagger(n)\sin\vartheta_n + c_\uparrow^\dagger(n+q)\cos\vartheta_n. \tag{8.35b}$$

In a single skyrmion ($q=1$) configuration only the nearest neighbouring sites (n and $n+1$) are mixed in $\xi(n)$. For $q=2$ the mixing appears among the sites with even n and separately among the ones with odd n, while no mixing appears among even and odd sites. In this sense there arise two separate chains of mixed sites where the above steps can be carried out separately. In the case of general q there are q separate chains, so nothing essentially new occurs.

8.4 Antiskyrmions

In the analogy with q-skyrmions the q-antiskyrmions are introduced as W-transformation of a q-electron state

$$|\mathfrak{S}_+^q\rangle = e^{iW^+}c_\downarrow^\dagger(0)c_\downarrow^\dagger(1)\cdots c_\downarrow^\dagger(q-1)|\mathrm{g}\rangle, \tag{8.36a}$$

$$iW^+ = \sum_n \theta_n\big[c_\downarrow^\dagger(n+q)c_\uparrow(n) - c_\uparrow^\dagger(n)c_\downarrow(n+q)\big], \tag{8.36b}$$

where $\theta_n \to 0$ is assumed for $n\to\infty$ thus guaranteeing the corresponding field averages approach the ground state values asymptotically.

Denoting

$$iW^+(n) = \theta_n\big[c_\downarrow^\dagger(n+q)c_\uparrow(n) - c_\uparrow^\dagger(n)c_\downarrow(n+q)\big], \tag{8.37}$$

we find

$$e^{iW^+} = e^{iW^+(0)+iW^+(1)+iW^+(2)+\cdots} = e^{iW^+(0)}e^{iW^+(1)}e^{iW^+(2)}\cdots, \tag{8.38}$$

as follows from $[W^+(m),W^+(n)]=0$. Besides, $W^+(n)$ comprise the down-spin creators only with $n\geqslant q$, and therefore they all commute with $c_\downarrow^\dagger(n)$ with $n=0,1,\ldots,q-1$. Consequently we can rewrite (8.36a) as

Then the state (8.36) can be written as

$$\begin{aligned}
|\mathfrak{S}_+^q\rangle &= c_\downarrow^\dagger(0)c_\downarrow^\dagger(1)\cdots c_\downarrow^\dagger(q-1)e^{iW^+}|\mathrm{g}\rangle = \\
&= c_\downarrow^\dagger(0)c_\downarrow^\dagger(1)\cdots c_\downarrow^\dagger(q-1)e^{iW^+}\Big[c_\uparrow^\dagger(0)c_\uparrow^\dagger(1)c_\uparrow^\dagger(2)\cdots\Big]|\varnothing\rangle = \\
&= c_\downarrow^\dagger(0)c_\downarrow^\dagger(1)\cdots c_\downarrow^\dagger(q-1)\Big\{\big[e^{iW^+}c_\uparrow^\dagger(0)e^{-iW^+}\big]\big[e^{iW^+}c_\uparrow^\dagger(1)e^{-iW^+}\big]\big[e^{iW^+}c_\uparrow^\dagger(2)e^{-iW^+}\big]\cdots\Big\}|\varnothing\rangle.
\end{aligned} \tag{8.39}$$

Employing the same trick as in skyrmion case we obtain

$$e^{iW^+}c_\uparrow^\dagger(n)e^{-iW^+} = e^{iW^+(n)}c_\uparrow^\dagger(n)e^{-iW^+(n)} = c_\uparrow^\dagger(n)\cos\theta_n + c_\downarrow^\dagger(n+q)\sin\theta_n, \tag{8.40}$$

hence

$$|\mathfrak{S}_+^q\rangle = c_\downarrow^\dagger(0)c_\downarrow^\dagger(1)\cdots c_\downarrow^\dagger(q-1)\Big\{\prod_{n=0}^{\infty}\big[c_\uparrow^\dagger(n)\cos\theta_n + c_\downarrow^\dagger(n+q)\sin\theta_n\big]\Big\}|\varnothing\rangle. \tag{8.41}$$

For the sake of further convenience we rewrite this in the form close to that of q-skyrmion given by (8.35). For this purpose we introduce

$$\zeta^\dagger(n) = c_\uparrow^\dagger(n-q)\cos\theta_{n-q} + c_\downarrow^\dagger(n)\sin\theta_{n-q}, \tag{8.42}$$

with convention $\cos\theta_n = 0$ for $n < 0$. Then the q-antiskyrmion state can be presented as

$$|\mathfrak{S}_+^q\rangle = \prod_{n=0}^{\infty}\zeta^\dagger(n)\cdot|\varnothing\rangle. \tag{8.43}$$

Recall the relation (6.20) stating that the number of extra particles brought by excitations is related to the topological charge of the corresponding NC Grassmannian field as $Q = -\Delta N$. Besides, as argued in section 5.1, amount of particles comprised in a Fock state is unaffected when W-transformation applied. Therefore, the amount extra particles (compared to that in the ground state) in skyrmion and antiskyrmion states are $\Delta N = -q$ and $\Delta N = +q$ respectively. Therefore we can predict that the corresponding NC Grassmannian will carry the topological charges $Q = +q$ and $Q = -q$ respectively. This point is considered in the subsequent sections.

8.5 Two-Point Averages

In order to construct the particle and isospin densities for skyrmion and antiskyrmion configurations, as well as the underlying NC CP fields, we need to calculate the averages $\langle\mathfrak{S}|c_\mu^\dagger(m)c_\nu(n)|\mathfrak{S}\rangle$ with respect to these states.

We present the detailed calculations for q-skyrmions. Let us first remark that

$$\{c_\mu(m),\xi^\dagger(n)\} = \{c_\mu(m),c_\downarrow^\dagger(n)\sin\vartheta_n + c_\uparrow^\dagger(n+q)\cos\vartheta_n\} =$$

$$= \delta_{\mu\downarrow}\delta_{mn}\sin\vartheta_n + \delta_{\mu\uparrow}\delta_{m,n+q}\cos\vartheta_n. \tag{8.44}$$

For $m < q$ we have $\{c_\uparrow(m),\xi^\dagger(n)\} = 0$ hence

$$c_\uparrow(m)|\mathfrak{S}_-^q\rangle = 0. \tag{8.45}$$

For $m \geqslant q$ we use (8.44) and proceed as follows

$$c_\uparrow(m)|\mathfrak{S}_-^q\rangle = c_\uparrow(m)\{\xi^\dagger(0)\xi^\dagger(1)\cdots\xi^\dagger(m-q-1)\xi^\dagger(m-q)\xi^\dagger(m-q+1)\cdots\}|\varnothing\rangle =$$

$$= (-1)^{m-q}\{\xi^\dagger(0)\xi^\dagger(1)\cdots\xi^\dagger(m-q-1)c_\uparrow(m)\xi^\dagger(m-q)\xi^\dagger(m-q+1)\cdots\}|\varnothing\rangle. \tag{8.46}$$

In accord with (8.44) we have

$$c_\uparrow(m)\xi^\dagger(m-q) = \cos\vartheta_{m-q} - \xi^\dagger(m-q)c_\uparrow(m), \tag{8.47}$$

and pass to

$$c_\uparrow(m)|\mathfrak{S}_-^q\rangle = (-1)^{m-q}\cos\vartheta_{m-q}\{\xi^\dagger(0)\xi^\dagger(1)\cdots\xi^\dagger(m-q-1)\xi^\dagger(m-q+1)\cdots\}|\varnothing\rangle -$$

$$-(-1)^{m-q}\{\xi^\dagger(0)\xi^\dagger(1)\cdots\xi^\dagger(m-q-1)\xi^\dagger(m-q)c_\uparrow(m)\xi^\dagger(m-q+1)\cdots\}|\varnothing\rangle, \tag{8.48}$$

where the second term vanishes since $c_\uparrow(m)$ commutes with all ξ-operators on the right and can be brought to $|\varnothing\rangle$. We thus obtain

$$c_\uparrow(m)|\mathfrak{S}_-^q\rangle = (-1)^{m-q}\cos\vartheta_{m-q}\{\xi^\dagger(0)\xi^\dagger(1)\cdots\xi^\dagger(m-q-1)\xi^\dagger(m-q+1)\cdots\}|\varnothing\rangle. \tag{8.49}$$

On the other hand, using $\{\xi(m),\xi^\dagger(n)\} = \delta_{mn}$ and $\xi(m)|\varnothing\rangle = 0$ we find

$$\xi(m-q)|\mathfrak{S}_-^q\rangle = (-1)^{m-q}\{\xi^\dagger(0)\xi^\dagger(1)\cdots\xi^\dagger(m-q-1)\xi^\dagger(m-q+1)\cdots\}|\varnothing\rangle. \tag{8.50}$$

Comparing this to (8.49) we find

$$c_\uparrow(m)|\mathfrak{S}_-^q\rangle = \cos\vartheta_{m-q}\xi(m-q)|\mathfrak{S}_-^q\rangle, \tag{8.51}$$

which can be extended to all m, thus embracing (8.45), with the convention $\cos\vartheta_{n<0} = 0$.

Performing similar manipulations for $c_\downarrow(m)|\mathfrak{S}_-^q\rangle$ we find

$$c_\downarrow(m)|\mathfrak{S}_-^q\rangle = \sin\vartheta_m \xi(m)|\mathfrak{S}_-^q\rangle. \tag{8.52}$$

Summarising, the averages $\langle\mathfrak{S}_-^q|c_\mu^\dagger(m)c_\nu(n)|\mathfrak{S}_-^q\rangle$ can be expressed in terms of $\langle\mathfrak{S}_-^q|\xi^\dagger(m)\xi(n)|\mathfrak{S}_-^q\rangle$ which due to $\{\xi^\dagger(m),\xi(n)\} = \delta_{mn}$ are given by $\langle\mathfrak{S}_-^q|\xi^\dagger(m)\xi(n)|\mathfrak{S}_-^q\rangle = \delta_{mn}$.

Then the only non-vanishing components of $\langle\mathfrak{S}_-^q|c_\mu^\dagger(m)c_\nu(n)|\mathfrak{S}_-^q\rangle$ appear as

$$\langle\mathfrak{S}_-^q|c_\uparrow^\dagger(m)c_\uparrow(n)|\mathfrak{S}_-^q\rangle = \delta_{mn}\cos^2\vartheta_{n-q}, \tag{8.53a}$$

$$\langle\mathfrak{S}_-^q|c_\uparrow^\dagger(m)c_\downarrow(n)|\mathfrak{S}_-^q\rangle = \delta_{m,n+q}\cos\vartheta_n\sin\vartheta_n, \tag{8.53b}$$

$$\langle\mathfrak{S}_-^q|c_\downarrow^\dagger(m)c_\uparrow(n)|\mathfrak{S}_-^q\rangle = \delta_{n,m+q}\cos\vartheta_m\sin\vartheta_m, \tag{8.53c}$$

$$\langle\mathfrak{S}_-^q|c_\downarrow^\dagger(m)c_\downarrow(n)|\mathfrak{S}_-^q\rangle = \delta_{mn}\sin^2\vartheta_n, \tag{8.53d}$$

where the convention $\cos\vartheta_{n<0} = 0$ is adopted.

Similar calculation for antiskyrmions lead to

$$\langle\mathfrak{S}_+^q|c_\uparrow^\dagger(m)c_\uparrow(n)|\mathfrak{S}_+^q\rangle = \delta_{mn}\cos^2\theta_n, \tag{8.54a}$$

$$\langle\mathfrak{S}_+^q|c_\uparrow^\dagger(m)c_\downarrow(n)|\mathfrak{S}_+^q\rangle = \delta_{n,m+q}\cos\theta_m\sin\theta_m \tag{8.54b}$$

$$\langle\mathfrak{S}_+^q|c_\downarrow^\dagger(m)c_\uparrow(n)|\mathfrak{S}_+^q\rangle = \delta_{m,n+q}\cos\theta_n\sin\theta_n \tag{8.54c}$$

$$\langle\mathfrak{S}_+^q|c_\downarrow^\dagger(m)c_\downarrow(n)|\mathfrak{S}_+^q\rangle = \delta_{mn}\sin^2\theta_{n-q} \tag{8.54d}$$

with the convention $\cos\theta_{n<0} = 0$.

8.6 Classical Fields

We calculate classical fields describing particle and spin densities corresponding to skyrmion and antiskyrmion states set by (8.35) and (8.43). For the physical fields we have

$$\rho_{\rm cl}(\boldsymbol{r}) = \sum_{mn}\langle\boldsymbol{r}|0,n\rangle\langle 0,m|\boldsymbol{r}\rangle\langle\mathfrak{S}_\pm^q|c_\mu^\dagger(m)c_\mu(n)|\mathfrak{S}_\pm^q\rangle, \tag{8.55a}$$

$$I_a^{\rm cl}(\boldsymbol{r}) = \tfrac{1}{2}(\sigma_a)_{\mu\nu}\sum_{mn}\langle\boldsymbol{r}|0,n\rangle\langle 0,m|\boldsymbol{r}\rangle\langle\mathfrak{S}_\pm^q|c_\mu^\dagger(m)c_\nu(n)|\mathfrak{S}_\pm^q\rangle. \tag{8.55b}$$

Using here (2.23) for $\langle\boldsymbol{r}|0,n\rangle$ together with (8.53) we find the skyrmion fields to be (with $x+iy = re^{+i\phi}$)

$$\rho_{\rm cl}(\boldsymbol{r}) = \frac{1}{2\pi\ell^2}e^{-\frac{1}{2}\ell^{-2}r^2}\sum_{n=0}^{\infty}\frac{\cos^2\vartheta_{n-q}+\sin^2\vartheta_n}{n!}\left[\frac{r^2}{2\ell^2}\right]^n, \tag{8.56a}$$

$$I_z^{\rm cl}(\boldsymbol{r}) = \frac{1}{4\pi\ell^2}e^{-\frac{1}{2}\ell^{-2}r^2}\sum_{n=0}^{\infty}\frac{\cos^2\vartheta_{n-q}-\sin^2\vartheta_n}{n!}\left[\frac{r^2}{2\ell^2}\right]^n, \tag{8.56b}$$

$$I_x^{\rm cl}(\boldsymbol{r}) = \frac{+\cos(q\phi)}{2\pi\ell^2}\left[\frac{r}{\sqrt{2}\ell}\right]^q e^{-\frac{1}{2}\ell^{-2}r^2}\sum_{n=0}^{\infty}\frac{\cos\vartheta_n\sin\vartheta_n}{\sqrt{n!(n+q)!}}\left[\frac{r^2}{2\ell^2}\right]^n, \tag{8.56c}$$

$$I_y^{\rm cl}(\boldsymbol{r}) = \frac{-\sin(q\phi)}{2\pi\ell^2}\left[\frac{r}{\sqrt{2}\ell}\right]^q e^{-\frac{1}{2}\ell^{-2}r^2}\sum_{n=0}^{\infty}\frac{\cos\vartheta_n\sin\vartheta_n}{\sqrt{n!(n+q)!}}\left[\frac{r^2}{2\ell^2}\right]^n. \tag{8.56d}$$

The same with (8.54) for antiskyrmions leads to

$$\rho_{\mathrm{cl}}(\boldsymbol{r}) = \frac{1}{2\pi\ell^2} e^{-\frac{1}{2}\ell^{-2}r^2} \sum_{n=0}^{\infty} \frac{\cos^2\theta_{n-q} + \sin^2\theta_n}{n!} \left[\frac{r^2}{2\ell^2}\right]^n, \tag{8.57a}$$

$$I_z^{\mathrm{cl}}(\boldsymbol{r}) = \frac{-1}{4\pi\ell^2} e^{-\frac{1}{2}\ell^{-2}r^2} \sum_{n=0}^{\infty} \frac{\cos^2\theta_{n-q} - \sin^2\theta_n}{n!} \left[\frac{r^2}{2\ell^2}\right]^n, \tag{8.57b}$$

$$I_x^{\mathrm{cl}}(\boldsymbol{r}) = \frac{+\cos(q\phi)}{2\pi\ell^2} \left[\frac{r}{\sqrt{2}\,\ell}\right]^q e^{-\frac{1}{2}\ell^{-2}r^2} \sum_{n=0}^{\infty} \frac{\cos\theta_n \sin\theta_n}{\sqrt{n!(n+q)!}} \left[\frac{r^2}{2\ell^2}\right]^n, \tag{8.57c}$$

$$I_y^{\mathrm{cl}}(\boldsymbol{r}) = \frac{+\sin(q\phi)}{2\pi\ell^2} \left[\frac{r}{\sqrt{2}\,\ell}\right]^q e^{-\frac{1}{2}\ell^{-2}r^2} \sum_{n=0}^{\infty} \frac{\cos\theta_n \sin\theta_n}{\sqrt{n!(n+q)!}} \left[\frac{r^2}{2\ell^2}\right]^n. \tag{8.57d}$$

Typical spin textures of skyrmion and antiskyrmion excitations are depicted in figure 8.2.

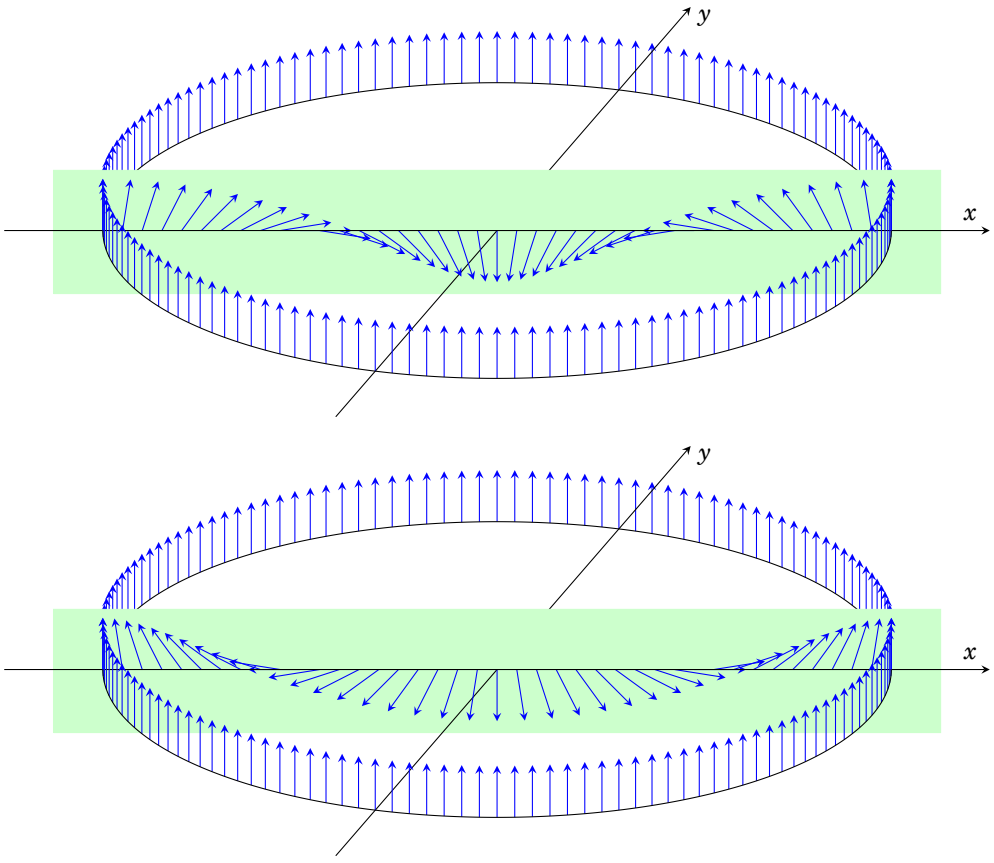


Fig. 8.2: Typical spin textures for skyrmion (upper panel) and antiskyrmion (lower panel) excitations.

8.7 Non-Commutative CP^1 Fields

In this section we construct the non-commutative CP^1 fields associated with skyrmion and antiskyrmion configurations. This is complex-valued two-component column

$$f(\boldsymbol{r}) = \begin{pmatrix} f_\uparrow(\boldsymbol{r}) \\ f_\downarrow(\boldsymbol{r}) \end{pmatrix}, \tag{8.58}$$

normalized as

$$f^\dagger(\boldsymbol{r}) \star f(\boldsymbol{r}) = 1, \tag{8.59}$$

and giving the desired core averages in accord with

$$\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r}) = \frac{1}{2\pi\ell^2} f_\mu(\boldsymbol{r}) \star f^*_\nu(\boldsymbol{r}). \tag{8.60}$$

We attempt to determine $f_\mu(\boldsymbol{r})$ provided $\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r})$ are known. This can be easily calculated in terms of Weyl operators defined by (3.6). If more precisely, let $O[f_\mu]$ and $O[\hat{D}^{\rm cl}_{\mu\nu}]$ be the Weyl operators corresponding to $f_\mu(\boldsymbol{r})$ and $\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{r})$. Then using (3.14) together with $O[f^*] = O^\dagger[f]$ we write the relations (8.59) and (8.60) in terms of Weyl operators as

$$O^\dagger[f_\mu] O[f_\mu] = \mathbb{I}, \tag{8.61a}$$

$$O[f_\mu] O^\dagger[f_\nu] = 2\pi\ell^2 O[\hat{D}^{\rm cl}_{\mu\nu}]. \tag{8.61b}$$

If we manage to calculate the Weyl operator $O[\hat{D}^{\rm cl}_{\mu\nu}]$ for skyrmion and antiskyrmion states, we'll be then able first to solve out $O[f_\mu]$ from (8.61) and subsequently to restore $f_\mu(\boldsymbol{r})$.

Employing the definition (3.6) of the Weyl operator we write

$$O[\hat{D}^{\rm cl}_{\mu\nu}] = \frac{1}{2\pi} \int \hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{k}) e^{+i\boldsymbol{k}\boldsymbol{X}} d\boldsymbol{k}, \tag{8.62}$$

and insert the identity operators $|m\rangle\langle m| = \mathbb{I}$ and $|n\rangle\langle n = \mathbb{I}$ in the right hand side

$$O[\hat{D}^{\rm cl}_{\mu\nu}] = \sum_m |m\rangle\langle m| \left[\frac{1}{2\pi} \int \hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{k}) e^{+i\boldsymbol{k}\boldsymbol{X}} d\boldsymbol{k} \right] \sum_n |n\rangle\langle n| =$$

$$= \sum_{mn} \left[\frac{1}{2\pi} \int \hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{k}) \langle m| e^{+i\boldsymbol{k}\boldsymbol{X}} |n\rangle d\boldsymbol{k} \right] |m\rangle\langle n|. \tag{8.63}$$

Substituting here

$$\hat{D}^{\rm cl}_{\mu\nu}(\boldsymbol{k}) = \frac{1}{2\pi} \sum_{st} \langle s| e^{-i\boldsymbol{k}\boldsymbol{X}} |t\rangle \langle \mathfrak{S}| c^\dagger_\nu(s) c_\mu(t) |\mathfrak{S}\rangle, \tag{8.64}$$

and using

$$\int \langle m| e^{-i\boldsymbol{k}\boldsymbol{X}} |n\rangle \langle s| e^{+i\boldsymbol{k}\boldsymbol{X}} |t\rangle d\boldsymbol{k} = 2\pi\ell^{-2} \delta_{mt} \delta_{ns}, \tag{8.65}$$

we come to

$$2\pi\ell^2 O[\hat{D}^{\rm cl}_{\mu\nu}] = \sum_{mn} \langle \mathfrak{S}| c^\dagger_\nu(n) c_\mu(m) |\mathfrak{S}\rangle \cdot |m\rangle\langle n|. \tag{8.66}$$

Hence, with quantities $\langle \mathfrak{S}^q_\pm | c^\dagger_\nu(n) c_\mu(m) | \mathfrak{S}^q_\pm \rangle$ already calculated we can explicitly construct the Weyl operator $O[\hat{D}^{\rm cl}_{\mu\nu}]$. We then solve out $O[f_\mu]$ from (8.61) in $|n\rangle$-basis and subsequently restore the non-commutative field $f_\mu(\boldsymbol{r})$ by use of the relation [*Harvey, hep-th/0102076*]

$$|n+\alpha\rangle\langle n| = 2^{\alpha+1} \frac{(-1)^n \sqrt{n!}}{\sqrt{(n+\alpha)!}} O\big[z^\alpha e^{-\ell^{-2} r^2} L^\alpha_n (2\ell^{-2} r^2) \big], \tag{8.67}$$

where $\sqrt{2} z \equiv \ell^{-1}(x+iy)$, and L^α_n is the generalized Laguerre polynomial.

Skyrmions

Employing (8.53) we find

$$2\pi\ell^2 O[\hat{D}^{\rm cl}_{\uparrow\uparrow}] = \sum_{n=0}^{\infty} \cos^2\vartheta_n |n+q\rangle\langle n+q|, \tag{8.68a}$$

$$2\pi\ell^2 O[\hat{D}^{\rm cl}_{\uparrow\downarrow}] = \sum_{n=0}^{\infty} \cos\vartheta_n \sin\vartheta_n |n+q\rangle\langle n|, \tag{8.68b}$$

$$2\pi\ell^2 O[\hat{D}^{\rm cl}_{\downarrow\uparrow}] = \sum_{n=0}^{\infty} \cos\vartheta_n \sin\vartheta_n |n\rangle\langle n+q|, \tag{8.68c}$$

$$2\pi\ell^2 O[\hat{D}^{\rm cl}_{\downarrow\downarrow}] = \sum_{n=0}^{\infty} \sin^2\vartheta_n |n\rangle\langle n|. \tag{8.68d}$$

Then the equations (8.61) are resolved by

$$O[f^-_{\uparrow}] = \sum_{n=0}^{\infty} \cos\vartheta_n |n+q\rangle\langle n|, \tag{8.69a}$$

$$O[f^-_{\downarrow}] = \sum_{n=0}^{\infty} \sin\vartheta_n |n\rangle\langle n|. \tag{8.69b}$$

Using (8.67) we find the corresponding non-commutative field in $\boldsymbol{r}$-space to be

$$f^-_{\uparrow}(\boldsymbol{r}) = 2e^{+iq\phi}(\sqrt{2}r/\ell)^q e^{-r^2/\ell^2} \sum_{n=0}^{\infty} \cos\vartheta_n \frac{(-1)^n\sqrt{n!}}{\sqrt{(n+q)!}} L^q_n(2\ell^{-2}r^2), \tag{8.70a}$$

$$f^-_{\downarrow}(\boldsymbol{r}) = 2e^{-r^2/\ell^2} \sum_{n=0}^{\infty} (-1)^n \sin\vartheta_n L_n(2r^2/\ell^2), \tag{8.70b}$$

or equivalently

$$f^-(\boldsymbol{r}) = 2e^{-r^2/\ell^2} \sum_{n=0}^{\infty} (-1)^n \left\{ \sin\vartheta_n L_n(2r^2/\ell^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{+iq\phi}(\sqrt{2}r/\ell)^q \cos\vartheta_n \frac{\sqrt{n!}}{\sqrt{(n+q)!}} L^q_n(2\ell^{-2}r^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}. \tag{8.71}$$

Antiskyrmions

In order to obtain the analogous expressions for antiskyrmions we use (8.54) which lead to

$$2\pi\ell^2 O[\hat{D}^{\rm cl}_{\uparrow\uparrow}] = \sum_{n=0}^{\infty} \cos^2\theta_n |n\rangle\langle n|, \tag{8.72a}$$

$$2\pi\ell^2 O[\hat{D}^{\rm cl}_{\uparrow\downarrow}] = \sum_{n=0}^{\infty} \cos\theta_n \sin\theta_n |n\rangle\langle n+q|, \tag{8.72b}$$

$$2\pi\ell^2 O[\hat{D}^{\rm cl}_{\downarrow\uparrow}] = \sum_{n=0}^{\infty} \cos\theta_n \sin\theta_n |n+q\rangle\langle n|, \tag{8.72c}$$

$$2\pi\ell^2 O[\hat{D}^{\rm cl}_{\downarrow\downarrow}] = \sum_{n=0}^{\infty} \sin^2\theta_{n-q} |n\rangle\langle n| = \sum_{n=0}^{q-1} |n\rangle\langle n| + \sum_{n=0}^{\infty} \sin^2\theta_n |n+q\rangle\langle n+q|, \tag{8.72d}$$

where the convention $\cos\theta_{n<0} = 0$, (*i.e.* $\sin\theta_{n<0} = 1$) has been employed.

These are resolved uniquely by

$$O[f^+_\uparrow] = \sum_{n=0}^{\infty} \cos\theta_n |n\rangle\langle n+q|, \tag{8.73a}$$

$$O[f^+_\downarrow] = \sum_{n=0}^{q-1} |n\rangle\langle n| + \sum_{n=0}^{\infty} \sin\theta_n |n+q\rangle\langle n+q|, \tag{8.73b}$$

which by use of (8.67) lead to

$$f^+_\uparrow(\boldsymbol{r}) = 2e^{-iq\phi}(\sqrt{2}r/\ell)^q e^{-\ell^{-2}r^2} \sum_{n=0}^{\infty} \cos\theta_n \frac{(-1)^n\sqrt{n!}}{\sqrt{(n+q)!}} L^q_n(2\ell^{-2}r^2), \tag{8.74a}$$

$$f^+_\downarrow(\boldsymbol{r}) = 2e^{-\ell^{-2}r^2}\Big\{\sum_{n=0}^{q-1}(-1)^n L_n(2\ell^{-2}r^2) + \sum_{n=0}^{\infty}(-1)^{n+q}\sin\theta_n L_{n+q}(2\ell^{-2}r^2)\Big\}. \tag{8.74b}$$

8.8 Topological Charge of Skyrmions and Antiskyrmions

Topological charge of a non-commutative CP^1-field has been derived in section 6 to be

$$Q = \frac{1}{2\pi\ell^2}\int \Big\{\bar{f}_\mu(\boldsymbol{r}) \star f_\mu(\boldsymbol{r}) - f_\mu(\boldsymbol{r}) \star \bar{f}_\mu(\boldsymbol{r})\Big\} d\boldsymbol{r}, \tag{8.75}$$

which in terms of the corresponding Weyl operators can be expressed as

$$Q = \mathtt{Tr}\Big\{O[\bar{f}_\mu]O[f_\mu] - O[f_\mu]O[\bar{f}_\mu]\Big\}. \tag{8.76}$$

Here we cannot use $\mathtt{Tr}([A,B]) = \mathtt{Tr}(AB) - \mathtt{Tr}(BA) = 0$, which is valid only when $\mathtt{Tr}(AB)$ and $\mathtt{Tr}(BA)$ are separately well defined. This is not the case here, *i.e.* the trace must be calculated after the commutator is resolved. Using (8.69) we obtain

$$Q^- = \mathtt{Tr}\Big\{\sum_{n=0}^{\infty}\cos^2\vartheta_n |n\rangle\langle n| - \sum_{n=0}^{\infty}\cos^2\vartheta_n |n+q\rangle\langle n+q|\Big\}. \tag{8.77}$$

This is inappropriate for calculating the trace, since the separate pieces are divergent due to the behaviour $\cos\vartheta_n \to 1$ as $n\to\infty$, as implied by the boundary condition (4.41a). Rewriting this expression in terms of $\sin\vartheta_n$ we obtain

$$Q^- = \mathtt{Tr}\Big\{|0\rangle\langle 0| + |1\rangle\langle 1| + \cdots + |q-1\rangle\langle q-1| - \sum_{n=0}^{\infty}\sin^2\vartheta_n\big[|n\rangle\langle n| - |n+q\rangle\langle n+q|\big]\Big\}, \tag{8.78}$$

where each piece is well defined since $\sin\vartheta_n \to 0$ as $n\to\infty$. Therefore, the traces can be calculated separately what eventually gives $Q^- = q$. Similar manipulations for antiskyrmions lead to $Q^+ = -q$. These agree with $Q = -\Delta N$.

8.9 Many-Body Wave Functions

We derive many-body wave functions for skyrmion and antiskyrmion states. We consider the system of $K+1$ available Landau sites ($n = 0,\ldots,K$). Ground state of this system at $\nu = 1$ is realized when each Landau site comprises by one electron, meaning $K+1$ electrons in total. The corresponding Fock is written as

$$|\mathrm{g}\rangle = c^\dagger_\uparrow(0)c^\dagger_\uparrow(1)\cdots c^\dagger_\uparrow(K)|\varnothing\rangle. \tag{8.79}$$

We start discussing the skyrmion state which is constructed by removing one electron from $|\mathrm{g}\rangle$ and then performing W-transformation. This appears as

$$|\mathfrak{S}_-\rangle = \prod_{n=0}^{K-1}\Big[c^\dagger_\downarrow(n)\sin\vartheta_n + c^\dagger_\uparrow(n+1)\cos\vartheta_n\Big]|\varnothing\rangle, \tag{8.80}$$

and in the limit of $\vartheta_n \to 0$ turns into the hole-excited state

$$|\mathfrak{S}_-\rangle \;\to\; c^\dagger_\uparrow(1)\cdots c^\dagger_\uparrow(K)|\varnothing\rangle = c_\uparrow(0)|\mathrm{g}\rangle. \tag{8.81}$$

Skyrmion comprises K electrons, and therefore we introduce the K-body quantum mechanical wave function as

$$\mathfrak{S}^{-}_{\mu_1\cdots\mu_K}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)=\langle\varnothing|\psi_{\mu_1}(\boldsymbol{r}_1)\cdots\psi_{\mu_K}(\boldsymbol{r}_K)|\mathfrak{S}_-\rangle, \tag{8.82}$$

where the electron field operator defined over $K+1$ Landau sites is given by

$$\psi_\mu(\boldsymbol{r})=(2\pi\ell^2)^{-\frac{1}{2}}e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{K}\alpha_n z^n c_\mu(n), \tag{8.83}$$

with $\alpha_n\equiv(n!)^{-\frac{1}{2}}$.

We first calculate the component with $\mu_1=\mu_2=\cdots=\mu_K=\uparrow$. It is given by

$$\mathfrak{S}^{-}_{\uparrow\uparrow\ldots\uparrow}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)=\langle\varnothing|\psi_\uparrow(\boldsymbol{r}_1)\cdots\psi_\uparrow(\boldsymbol{r}_K)\prod_{n=0}^{K-1}\Big[\sin\vartheta_n c^\dagger_\downarrow(n)+\cos\vartheta_n c^\dagger_\uparrow(n+1)\Big]|\varnothing\rangle. \tag{8.84}$$

Remark that in this expression there are K up-spin annihilation operators standing on the left. Therefore, there must be the same amount of up-spin creation operators on the right, otherwise the matrix element vanishes. Consequently, the terms "$\sin\vartheta_n c^*_\downarrow(n)$" in square brace can be all dropped and we pass to

$$\mathfrak{S}^{-}_{\uparrow\uparrow\ldots\uparrow}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)=\left\{\prod_{n=0}^{K-1}\cos\vartheta_n\right\}\langle\varnothing|\psi_\uparrow(\boldsymbol{r}_1)\cdots\psi_\uparrow(\boldsymbol{r}_K)c^\dagger_\uparrow(K)\cdots c^\dagger_\uparrow(2)c^\dagger_\uparrow(1)|\varnothing\rangle. \tag{8.85}$$

Writing $\psi_\uparrow(\boldsymbol{r})$ explicitly we pass to

$$\begin{aligned}\mathfrak{S}^{-}_{\uparrow\uparrow\ldots\uparrow}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)=&\left\{\prod_{n=0}^{K-1}\cos\vartheta_n\right\}(2\pi\ell^2)^{-\frac{K}{2}}e^{-\frac{1}{2}(|z_1|^2+|z_2|^2+\cdots+|z_K|^2)}\times\\
&\times\langle\varnothing|\big[\alpha_0c_\uparrow(0)+\alpha_1z_1c_\uparrow(1)+\alpha_2z_1^2c_\uparrow(2)+\cdots+\alpha_Kz_1^Kc_\uparrow(K)\big]\times\\
&\times\big[\alpha_0c_\uparrow(0)+\alpha_1z_2c_\uparrow(1)+\alpha_2z_2^2c_\uparrow(2)+\cdots+\alpha_Kz_2^Kc_\uparrow(K)\big]\times\cdots\\
&\cdots\times\big[\alpha_0c_\uparrow(0)+\alpha_1z_Kc_\uparrow(1)+\alpha_2z_K^2c_\uparrow(2)+\cdots+\alpha_Kz_K^Kc_\uparrow(K)\big]c^\dagger_\uparrow(K)\cdots c^\dagger_\uparrow(2)c^\dagger_\uparrow(1)|\varnothing\rangle.\end{aligned} \tag{8.86}$$

The terms $\alpha_0c_\uparrow(0)$ give no contribution since there is no $c^\dagger_\uparrow(0)$ operator among $c^\dagger_\uparrow(K)\cdots c^\dagger_\uparrow(2)c^\dagger_\uparrow(1)$. Dropping these terms and using the anticommutation relations of electron operators, we express the vacuum matrix element as

$$\begin{aligned}&\langle\varnothing|\big[\alpha_1z_1c_\uparrow(1)+\alpha_2z_1^2c_\uparrow(2)+\cdots+\alpha_Kz_1^Kc_\uparrow(K)\big]\times\\
&\times\big[\alpha_1z_2c_\uparrow(1)+\alpha_2z_2^2c_\uparrow(2)+\cdots+\alpha_Kz_2^Kc_\uparrow(K)\big]\times\cdots\\
&\cdots\times\big[\alpha_1z_Kc_\uparrow(1)+\alpha_2z_K^2c_\uparrow(2)+\cdots+\alpha_Kz_K^Kc_\uparrow(K)\big]c^\dagger_\uparrow(K)\cdots c^\dagger_\uparrow(2)c^\dagger_\uparrow(1)|\varnothing\rangle=\begin{vmatrix}\alpha_1z_1&\alpha_2z_1^2&\cdots&\alpha_Kz_1^K\\ \alpha_1z_2&\alpha_2z_2^2&\cdots&\alpha_Kz_2^K\\ \vdots&\vdots&\ddots&\vdots\\ \alpha_1z_K&\alpha_2z_K^2&\cdots&\alpha_Kz_K^K\end{vmatrix}.\end{aligned} \tag{8.87}$$

Employing the elementary rules of determinant calculus we eventually come to

$$\mathfrak{S}^{-}_{\uparrow\uparrow\ldots\uparrow}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)=C_K\cdot(z_1z_2\cdots z_K)\mathfrak{S}_0(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K), \tag{8.88}$$

where $\mathfrak{S}_0(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)$ is the Slater determinant

$$\mathfrak{S}_0(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)=\begin{vmatrix}1&z_1&\cdots&z_1^{K-1}\\ 1&z_2&\cdots&z_2^{K-1}\\ \vdots&\vdots&\ddots&\vdots\\ 1&z_K&\cdots&z_K^{K-1}\end{vmatrix}e^{-\frac{1}{2}(|z_1|^2+|z_2|^2+\cdots+|z_K|^2)}, \tag{8.89}$$

and C_K is given by

$$C_K = (2\pi\ell^2)^{-\frac{K}{2}}(\alpha_1\alpha_2\cdots\alpha_K)\prod_{n=0}^{K-1}\cos\vartheta_n. \tag{8.90}$$

We now consider the component with $\mu_1=\mu_2=\cdots=\mu_K=\downarrow$. It is given by

$$\mathfrak{S}^-_{\downarrow\downarrow\cdots\downarrow}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K) = \langle\varnothing|\psi_\downarrow(\boldsymbol{r}_1)\cdots\psi_\downarrow(\boldsymbol{r}_K)\prod_{n=0}^{K-1}\Big[\sin\vartheta_n c^\dagger_\downarrow(n)+\cos\vartheta_n c^\dagger_\uparrow(n+1)\Big]|\varnothing\rangle, \tag{8.91}$$

where the pieces "$\cos\vartheta_n c^\dagger_\uparrow(n+1)$" can be dropped since the wave operators $\psi_\downarrow(z)$ comprise only down-spin annihilators. Writing $\psi_\downarrow(z)$ explicitly we find

$$\begin{aligned}\mathfrak{S}^-_{\downarrow\downarrow\cdots\downarrow}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K) = \left\{\prod_{n=0}^{K-1}\sin\vartheta_n\right\}(2\pi\ell^2)^{-\frac{K}{2}}e^{-\frac{1}{2}(|z_1|^2+|z_2|^2+\cdots+|z_K|^2)}\times\\ \times\langle\varnothing|\big[\alpha_0c_\downarrow(0)+\alpha_1z_1c_\downarrow(1)+\alpha_2z_1^2c_\downarrow(2)+\cdots+\alpha_Kz_1^Kc_\downarrow(K)\big]\times\\ \times\big[\alpha_0c_\downarrow(0)+\alpha_1z_2c_\downarrow(1)+\alpha_2z_2^2c_\downarrow(2)+\cdots+\alpha_Kz_2^Kc_\downarrow(K)\big]\times\cdots\\ \cdots\times\big[\alpha_0c_\downarrow(0)+\alpha_1z_Kc_\downarrow(1)+\alpha_2z_K^2c_\downarrow(2)+\cdots+\alpha_Kz_K^Kc_\downarrow(K)\big]c^\dagger_\downarrow(K-1)\cdots c^\dagger_\downarrow(1)c^\dagger_\downarrow(0)|\varnothing\rangle.\end{aligned} \tag{8.92}$$

Provided there is no $c^\dagger_\downarrow(K)$ operator among $c^\dagger_\downarrow(K-1)\cdots c^\dagger_\downarrow(1)c^\dagger_\downarrow(0)$, the terms $\alpha_Kz_K^Kc_\downarrow(K)$ can be dropped in square braces. Then the matrix elements with respect to the Fock vacuum can be calculated to be

$$\begin{aligned}&\langle\varnothing|\big[\alpha_0z_1^0c_\downarrow(0)+\cdots+\alpha_{K-1}z_1^{K-1}c_\downarrow(K-1)\big]\times\\ &\times\big[\alpha_0z_2^0c_\downarrow(0)+\cdots+\alpha_{K-1}z_2^{K-1}c_\downarrow(K-1)\big]\times\cdots\\ &\cdots\times\big[\alpha_0z_K^0c_\downarrow(0)+\cdots+\alpha_{K-1}z_K^{K-1}c_\downarrow(K-1)\big]c^\dagger_\downarrow(K-1)\cdots c^\dagger_\downarrow(0)|\varnothing\rangle = \begin{vmatrix}\alpha_0z_1^0 & \alpha_1z_1^1 & \cdots & \alpha_{K-1}z_1^{K-1}\\ \alpha_0z_2^0 & \alpha_1z_2^1 & \cdots & \alpha_{K-1}z_2^{K-1}\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_0z_K^0 & \alpha_1z_K^1 & \cdots & \alpha_{K-1}z_K^{K-1}\end{vmatrix},\end{aligned} \tag{8.93}$$

and after the similar determinant calculus we stop at

$$\mathfrak{S}^-_{\downarrow\downarrow\cdots\downarrow}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K) = C_K\cdot(\kappa_1\kappa_2\cdots\kappa_K)\mathfrak{S}_0(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K), \tag{8.94}$$

where

$$\kappa_n = \sqrt{n}\,\tan\vartheta_{n-1}. \tag{8.95}$$

Comparing (8.94) to (8.88) we find that $\mathfrak{S}^-_{\downarrow\downarrow\cdots\downarrow}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)$ differs from $\mathfrak{S}^-_{\uparrow\uparrow\cdots\uparrow}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)$ by $(\kappa_1\kappa_2\cdots\kappa_K)$ standing instead of $(z_1z_2\cdots z_K)$. Analogously, the wave function with "$\downarrow$" standing at s'th place can be obtained by replacing z_s by κ_s in $(z_1z_2\cdots z_K)$. Therefore, the skyrmion K-body wave function appears as

$$\mathfrak{S}^-_{\mu_1\cdots\mu_K}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K) = C_Ke^{-\frac{1}{2}(|z_1|^2+\cdots+|z_K|^2)}\begin{vmatrix}\begin{bmatrix}z_1\\ \kappa_1\end{bmatrix}_{\mu_1} & z_1\begin{bmatrix}z_1\\ \kappa_2\end{bmatrix}_{\mu_2} & \cdots & z_1^{K-1}\begin{bmatrix}z_1\\ \kappa_K\end{bmatrix}_{\mu_K}\\ \begin{bmatrix}z_2\\ \kappa_1\end{bmatrix}_{\mu_1} & z_2\begin{bmatrix}z_2\\ \kappa_2\end{bmatrix}_{\mu_2} & \cdots & z_2^{K-1}\begin{bmatrix}z_2\\ \kappa_K\end{bmatrix}_{\mu_K}\\ \vdots & \vdots & \ddots & \vdots\\ \begin{bmatrix}z_K\\ \kappa_1\end{bmatrix}_{\mu_1} & z_K\begin{bmatrix}z_K\\ \kappa_2\end{bmatrix}_{\mu_2} & \cdots & z_K^{K-1}\begin{bmatrix}z_K\\ \kappa_K\end{bmatrix}_{\mu_K}\end{vmatrix}. \tag{8.96}$$

We now turn to antiskyrmions which are constructed by adding one electron to $|\mathrm{g}\rangle$ and then applying W-transformation thus giving

$$|\mathfrak{S}_+\rangle = c^\dagger_\downarrow(0)\left\{\prod_{n=0}^{K-1}\Big[c^\dagger_\uparrow(n)\cos\theta_n+c^\dagger_\downarrow(n+1)\sin\theta_n\Big]\right\}c^\dagger_\uparrow(K)|\varnothing\rangle, \tag{8.97}$$

which in the limit of $\theta_n \to 0$ turns into the electron-excited state

$$|\mathfrak{S}_+\rangle \;\to\; c^\dagger_\downarrow(0)c^\dagger_\uparrow(0)\cdots c^\dagger_\uparrow(K)|\varnothing\rangle = c^\dagger_\downarrow(0)|\mathrm{g}\rangle. \tag{8.98}$$

Provided antiskyrmion comprises $K+2$ electrons, we introduce the $(K+2)$-body wave function

$$\mathfrak{S}^+_{\mu_1\cdots\mu_{K+2}}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_{K+2}) = \langle\varnothing|\psi_{\mu_1}(\boldsymbol{r}_1)\cdots\psi_{\mu_{K+2}}(\boldsymbol{r}_{K+2})|\mathfrak{S}_+\rangle. \tag{8.99}$$

Dealing with the lowest Landau levels it is a common opinion that apart the factor of $e^{-\frac{1}{2}(|z_1|^2+|z_2|^2+\cdots+|z_K|^2)}$ every possible many-body wave function must depend on z_s only, but not on $\bar{z}_s$, *e.g.* (8.96).

However, the antiskyrmion state (8.99) does not lead to a wave function with any reasonable structure like (8.96). The expression for general K is technically difficult to write down. As an example we present the one for $K=2$. Up to certain multiplicative factors the corresponding 4-body wave function appears as

$$\mathfrak{S}^+_{\uparrow\uparrow\uparrow\uparrow} = \mathfrak{S}^+_{\downarrow\downarrow\downarrow\downarrow} = 0\,, \tag{8.100a}$$

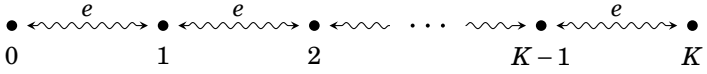
$$\mathfrak{S}^+_{\downarrow\uparrow\uparrow\uparrow} = (z_2-z_3)(z_2-z_4)(z_3-z_4)\,, \tag{8.100b}$$

$$\mathfrak{S}^+_{\downarrow\downarrow\uparrow\uparrow} = (z_1-z_2)(z_3-z_4)\big[z_3z_4-(z_1+z_2)(z_3+z_4)\big]\,, \tag{8.100c}$$

$$\mathfrak{S}^+_{\downarrow\downarrow\downarrow\uparrow} = (z_1-z_2)(z_1-z_3)(z_2-z_3)z_4^2\,, \tag{8.100d}$$

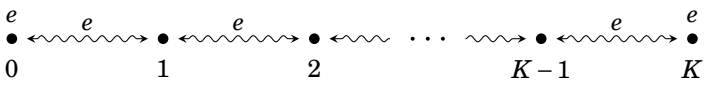
where no reasonable order is seen.

In this respect skyrmions and antiskyrmions might be regarded to possess rather different properties. Actually, this is not so, since the above scheme of dealing with antiskyrmions is not quite satisfactory. In order to clarify this issue let us first carry out a comment on the skyrmion state (8.79). This state can be graphically represented as



where the symbol $\leftsquigarrow$ denotes the ξ-particle *i.e.* an electron shared by two neighbouring sites. We call such an electron the site-transferable electron, since by manipulating with ϑ_n it can be transferred from one site to another. Thus, the skyrmion state comprises K site-transferable electrons which in the many-body wave function (8.96) are described in terms of K complex arguments.

Consider now the antiskyrmion state from the same point of view. The many-body wave function (8.97) comprises $K+2$ arguments which at first sight seem to correspond to the number of electrons in the Fock state (8.99). On the other hand, the graphical image of antiskyrmion appears as

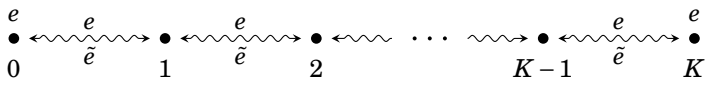


where two electrons out of total $K+2$ are pinned to $n=0$ and $n=K$, *i.e.* cannot be transferred to neighboring sites. So, though antiskyrmion is built up of $K+2$ electrons, it comprises only K site-transferrable electrons. Thus, skyrmion and antiskyrmion Fock states comprise the equal number of the degrees of freedom, since the two pinned electrons carry no degrees of freedom at all. However, antiskyrmion wave function defined by (8.99) involves $K+2$ complex arguments, which does not match the actual number of degrees of freedom. Nothing is wrong here, and the answer to this mismatch is that the components of antiskyrmion many-body wave function are not independent, but subject to certain functional relations. The most transparent manifestation of this statement are the relations (8.100a). Though the approach is not wrong by itself, it is inconvenient to work with. All these difficulties disappear if we develop a dual picture based on electron-hole symmetry. So far we used to construct the antiskyrmion Fock state by creating electrons over the vacuum state. Dual picture deals with the same state via removing electrons from completely filled system of $K+1$ sites.

Fock state of completely filled system is named as dual vacuum and we denote it as $|\tilde{\varnothing}\rangle$. It is given by

$$|\tilde{\varnothing}\rangle = \Big[\prod_{n=0}^{K} c^\dagger_\uparrow(n)\Big]\Big[\prod_{n=0}^{K} c^\dagger_\downarrow(n)\Big]|\varnothing\rangle. \tag{8.101}$$

From this state we should remove the correct degrees of freedom so that the remnant ones would be exactly what are comprised in the initial antiskyrmion state. In order to be clear we redraw the pictorial representation of antiskyrmion state in the following way



where the symbols $\tilde{e}$ denote electrons which should be added for filling out all vacancies. Say, there is one electron shared by $n=1$ and $n=2$, *i.e.* these two sites have one vacancy, which is "orthogonal" to what is already occupied. The shared electron is described by $\zeta^\dagger(1)=c_\uparrow^\dagger(1)\cos\theta_1+c_\downarrow^\dagger(2)\sin\theta_1$. In order to fill out the remnant vacancy shared by $n=1$ and $n=2$, we should use the operator $\tilde{\zeta}(1)$ which is "orthogonal" to $\zeta(1)$, *i.e.* we should have $\{\zeta(1),\tilde{\zeta}^\dagger(1)\}=0$. From this restriction we find $\tilde{\zeta}(1)=c_\uparrow(1)\sin\vartheta_1-c_\downarrow(2)\cos\vartheta_1$. Repeating similar considerations we find that antiskyrmion state can be brought up to the completely filled state by applying $\tilde{\zeta}^\dagger(0)\tilde{\zeta}^\dagger(1)\cdots\tilde{\zeta}^\dagger(K-1)$ where

$$\tilde{\zeta}(n)=c_\uparrow(n)\sin\vartheta_n-c_\downarrow(n+1)\cos\vartheta_n. \tag{8.102}$$

Therefore, from the point of view of dual ground state we express the antiskyrmion state (8.97) as

$$|\mathfrak{S}_+\rangle=\prod_{n=0}^{K-1}\left[c_\uparrow(n)\sin\vartheta_n-c_\downarrow(n+1)\cos\vartheta_n\right]|\tilde{\varnothing}\rangle. \tag{8.103}$$

The state $|\varnothing\rangle$ has been initially referred to as "vacuum" since it contains *no electrons*. In this connection the more appropriate name would be "electron vacuum", while the state $|\tilde{\varnothing}\rangle$ would be named as "hole vacuum" signifying that it contains no holes since being completely filled by electrons. In this light we introduce the hole annihilation $\tilde{c}_\mu(n)$ and creation $\tilde{c}_\mu^\dagger(n)$ operators as

$$\tilde{c}_{\uparrow\downarrow}(n)=c_{\downarrow\uparrow}^\dagger(n), \tag{8.104a}$$

$$\tilde{c}_{\uparrow\downarrow}^\dagger(n)=c_{\downarrow\uparrow}(n), \tag{8.104b}$$

where the spin indices are interchanged: annihilating an up-aligned spin we thus create the down-aligned one. These operators satisfy the standard anticommutation relation

$$\left\{\tilde{c}_\mu(m),\tilde{c}_\nu^\dagger(n)\right\}=\delta_{\mu\nu}\delta_{mn}, \tag{8.105}$$

and the hole vacuum is annihilated as

$$\tilde{c}_\mu(m)|\tilde{\varnothing}\rangle=0. \tag{8.106}$$

Now the antiskyrmion Fock state (8.103) can be rewritten as

$$|\mathfrak{S}_+\rangle=\prod_{n=0}^{K-1}\left[\tilde{c}_\downarrow^\dagger(n)\sin\theta_n-\tilde{c}_\uparrow^\dagger(n+1)\cos\theta_n\right]|\tilde{\varnothing}\rangle, \tag{8.107}$$

where the analogy with skyrmion state (8.80) is manifest.

In response to (8.104) we introduce the hole field operators $\tilde{\psi}_{\uparrow\downarrow}(\boldsymbol{r})=\psi_{\downarrow\uparrow}^\dagger(\boldsymbol{r})$ or equivalently

$$\tilde{\psi}_\mu(\boldsymbol{r})=(2\pi\ell^2)^{-\frac{1}{2}}e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{K}\alpha_n\bar{z}^n\tilde{c}_\mu(n), \tag{8.108}$$

where $\sqrt{2}\bar{z}=\ell^{-1}(x-iy)$.

Now the antiskyrmion many-body wave functions is defined as

$$\mathfrak{S}_{\mu_1\cdots\mu_K}^+(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)=\langle\tilde{\varnothing}|\tilde{\psi}_{\mu_1}(\boldsymbol{r}_1)\cdots\tilde{\psi}_{\mu_K}(\boldsymbol{r}_K)|\mathfrak{S}_+\rangle, \tag{8.109}$$

and contains K complex variables corresponding to K holes.Thus the nonphysical degrees of freedom associated with two nontransferable electrons do not appear at all, since those two electrons are accounted within the hole ground state $|\tilde{\varnothing}\rangle$.

Using the resemblance between (8.107) and (8.80) it is straightforward to write down the K-body wave function (8.109) once the expression (8.96) is already obtained. We thus have

$$\mathfrak{S}^{+}_{\mu_1\cdots\mu_K}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)=\tilde{C}_K e^{-\frac{1}{2}(|z_1|^2+\cdots+|z_K|^2)}\begin{vmatrix} \begin{bmatrix}\bar{z}_1\\ \tilde{\kappa}_1\end{bmatrix}_{\mu_1} & \bar{z}_1\begin{bmatrix}\bar{z}_1\\ \tilde{\kappa}_2\end{bmatrix}_{\mu_2} & \cdots & \bar{z}_1^{K-1}\begin{bmatrix}\bar{z}_1\\ \tilde{\kappa}_K\end{bmatrix}_{\mu_K}\\ \begin{bmatrix}\bar{z}_2\\ \tilde{\kappa}_1\end{bmatrix}_{\mu_1} & \bar{z}_2\begin{bmatrix}\bar{z}_2\\ \tilde{\kappa}_2\end{bmatrix}_{\mu_2} & \cdots & \bar{z}_2^{K-1}\begin{bmatrix}\bar{z}_2\\ \tilde{\kappa}_K\end{bmatrix}_{\mu_K}\\ \vdots & \vdots & \ddots & \vdots\\ \begin{bmatrix}\bar{z}_K\\ \tilde{\kappa}_1\end{bmatrix}_{\mu_1} & \bar{z}_K\begin{bmatrix}\bar{z}_K\\ \tilde{\kappa}_2\end{bmatrix}_{\mu_2} & \cdots & \bar{z}_K^{K-1}\begin{bmatrix}\bar{z}_K\\ \tilde{\kappa}_K\end{bmatrix}_{\mu_K}\end{vmatrix}, \tag{8.110}$$

where

$$\tilde{C}_K=(-1)^K(2\pi\ell^2)^{-\frac{K}{2}}(\alpha_1\alpha_2\cdots\alpha_K)\prod_{n=0}^{K-1}\cos\theta_n, \tag{8.111}$$

and

$$\tilde{\kappa}_n=-\sqrt{n}\,\tan\theta_{n-1}. \tag{8.112}$$

In final we comment on special type of skyrmions with the scale parameters set by: $\kappa_n=\kappa$ and $\tilde{\kappa}_n=\tilde{\kappa}$, or equivalently

$$\tan\vartheta_n=\frac{\kappa}{\sqrt{n+1}}, \tag{8.113a}$$

$$\tan\theta_n=\frac{-\tilde{\kappa}}{\sqrt{n+1}}. \tag{8.113b}$$

In that case the K-body wave functions (8.96) and (8.110) are reduced to

$$\mathfrak{S}^{-}_{\mu_1\cdots\mu_K}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)=C_K\left\{\prod_{n=1}^{K}\begin{bmatrix}z_n\\ \kappa\end{bmatrix}_{\mu_n}\right\}\mathfrak{S}_0(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K), \tag{8.114a}$$

$$\mathfrak{S}^{+}_{\mu_1\cdots\mu_K}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K)=\tilde{C}_K\left\{\prod_{n=1}^{K}\begin{bmatrix}\bar{z}_n\\ \tilde{\kappa}\end{bmatrix}_{\mu_n}\right\}\mathfrak{S}_0(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_K), \tag{8.114b}$$

where the spin part is factorized, and we name these configuration as "factorizable skyrmions".

In the next section we consider the factorizable skyrmions more in details in connection with the hard-core model.

Problems

(8.a) Derive (8.8) from (8.6).

(8.b) Derive (8.9) from (8.8).

(8.c) Derive (8.11) from (8.6).

(8.d) Employ (8.11) and derive (8.12).

(8.e) Verify that (8.69) substituted into (8.61b) lead to (8.68); verify the normalization (8.61a).

(8.f) Verify that (8.73) substituted into (8.61b) lead to (8.72); verify the normalization (8.61a).

Chapter 9

HARD-CORE AND COULOMB INTERACTIONS

Skyrmions contain infinitely many parameters ϑ_n and θ_n which are to be fixed to minimize the energy of the state. In the given subsection we carry out this programme for the hard-core interaction

$$V(\boldsymbol{r}_1-\boldsymbol{r}_2)=\delta(\boldsymbol{r}_1-\boldsymbol{r}_2), \tag{9.1}$$

and subsequently briefly comment on Coulomb interaction

$$V(\boldsymbol{r}_1-\boldsymbol{r}_2)=\frac{\alpha}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}. \tag{9.2}$$

Also, we will include the Zeeman term.

9.1 Hard-Core Interaction

In the case of a hard-core interaction the Hamiltonian (7.1) reads

$$H_e=\tfrac{1}{2}\int\big[\rho(\boldsymbol{r})-\rho_e\big]\big[\rho(\boldsymbol{r})-\rho_e\big]d\boldsymbol{r}. \tag{9.3}$$

Remind that ρ_e serves for electric neutrality of the system in the ground state. Therefore the quantity $\mathcal{N}_{\rm g}\equiv\int\rho_e d\boldsymbol{r}$ coincides with total number of particles in the ground state. On the other hand, the quantity $\mathcal{N}\equiv\int\rho(\boldsymbol{r})d\boldsymbol{r}$ is the operator counting the number of particles in a Fock state. Besides we consider the system at $\nu=1$ *i.e.* $\rho_e=(2\pi\ell^2)^{-1}$. We then rewrite (9.3) as

$$H_e=\tfrac{1}{2}\int\rho(\boldsymbol{r})\rho(\boldsymbol{r})d\boldsymbol{r}-\frac{1}{2\pi\ell^2}\mathcal{N}+\frac{1}{4\pi\ell^2}\mathcal{N}_{\rm g}=$$

$$=\tfrac{1}{2}\int\big[\psi_\uparrow^\dagger(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r})+\psi_\downarrow^\dagger(\boldsymbol{r})\psi_\downarrow(\boldsymbol{r})\big]\big[\psi_\uparrow^\dagger(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r})+\psi_\downarrow^\dagger(\boldsymbol{r})\psi_\downarrow(\boldsymbol{r})\big]d\boldsymbol{r}-\frac{1}{2\pi\ell^2}\mathcal{N}+\frac{1}{4\pi\ell^2}\mathcal{N}_{\rm g}. \tag{9.4}$$

Writing out the density operators explicitly $\rho=\psi_\uparrow^\dagger\psi_\uparrow+\psi_\downarrow^\dagger\psi_\downarrow$ and using here $\psi_\uparrow(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r})=\psi_\downarrow(\boldsymbol{r})\psi_\downarrow(\boldsymbol{r})=0$ together with

$$\big\{\psi_\mu(\boldsymbol{r}),\psi_\nu^\dagger(\boldsymbol{r})\big\}=\frac{\delta_{\mu\nu}}{2\pi\ell^2}, \tag{9.5}$$

we come to

$$H_e=\int\psi_\uparrow^\dagger(\boldsymbol{r})\psi_\downarrow^\dagger(\boldsymbol{r})\psi_\downarrow(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r})d\boldsymbol{r}-\frac{1}{4\pi\ell^2}(\mathcal{N}-\mathcal{N}_{\rm g}). \tag{9.6}$$

In terms of creation and annihilation operators this appears as

$$H_e=2\sum_{mnij}V_{mnij}c_\uparrow^\dagger(m)c_\downarrow^\dagger(i)c_\downarrow(j)c_\uparrow(n)-\frac{1}{4\pi\ell^2}(\mathcal{N}-\mathcal{N}_{\rm g}). \tag{9.7}$$

Fourier transform of the potential is given by $V(\boldsymbol{k})=(2\pi)^{-1}$. Then the matrix elements V_{mnst} can be calculated via (7.8) and by use of (2.19.14.6) [*Prudnikov, vol.2*] to give

$$V_{mnst}=\frac{1}{8\pi\ell^2}\frac{\sqrt{(m+s)!}}{\sqrt{m!s!}}\frac{\sqrt{(n+t)!}}{\sqrt{n!t!}}\frac{\delta_{m+s,n+t}}{\sqrt{2^{m+s+n+t}}}. \tag{9.8}$$

We now calculate the matrix element of (9.7) with respect to k-skyrmion Fock state. Using the relations (8.51) and (8.52) we write

$$\langle \mathfrak{S}^q_- | c^\dagger_\uparrow(m) c^\dagger_\downarrow(i) c_\downarrow(j) c_\uparrow(n) | \mathfrak{S}^q_- \rangle = \cos\vartheta_{m-q} \sin\vartheta_i \sin\vartheta_j \cos\vartheta_{n-q} \langle \mathfrak{S}^q_- | \xi^\dagger(m-q)\xi^\dagger(i)\xi(j)\xi(n-q) | \mathfrak{S}^q_- \rangle =$$

$$= \cos\vartheta_{m-q} \sin\vartheta_i \sin\vartheta_j \cos\vartheta_{n-q} (\delta_{mn}\delta_{ij} - \delta_{m,j+q}\delta_{n,i+q}). \tag{9.9}$$

We then obtain

$$\langle \mathfrak{S}^q_- | H_e | \mathfrak{S}^q_- \rangle = \frac{1}{4\pi\ell^2} \sum_{mn} \frac{(m+n)!}{m!n!} \frac{1}{2^{m+n}} \cos^2\vartheta_{m-q} \sin^2\vartheta_n -$$

$$- \frac{1}{4\pi\ell^2} \sum_{mn} \frac{\sqrt{(m+n+q)!}}{\sqrt{(m+q)!n!}} \frac{\sqrt{(m+n+q)!}}{\sqrt{(n+q)!m!}} \frac{1}{2^{m+n+q}} \cos\vartheta_m \sin\vartheta_n \sin\vartheta_m \cos\vartheta_n +$$

$$+ \frac{q}{4\pi\ell^2}, \tag{9.10}$$

where the last term reflects the extra number of electrons in q-skyrmion state to be $\langle \mathfrak{S}^q_- | \mathcal{N} | \mathfrak{S}^q_- \rangle - \mathcal{N}_{\rm g} = -q$.

Recalling the convention $\cos\vartheta_m = 0$ for $m < 0$, we first shift the summation index m as $m \to m+q$ and then rearrange into

$$\langle \mathfrak{S}^q_- | H_e | \mathfrak{S}^q_- \rangle = \frac{1}{8\pi\ell^2} \sum_{mn} \frac{(m+n+q)!}{2^{m+n+q} m!n!} \left[\frac{\sqrt{m!}}{\sqrt{(m+q)!}} \cos\vartheta_m \sin\vartheta_n - \frac{\sqrt{n!}}{\sqrt{(n+q)!}} \cos\vartheta_n \sin\vartheta_m \right]^2 + \frac{q}{4\pi\ell^2}. \tag{9.11}$$

First term of this expression being non-negative takes the minimal value for

$$\frac{\sqrt{m!}}{\sqrt{(m+q)!}} \frac{\cos\vartheta_m}{\sin\vartheta_m} = \frac{\sqrt{n!}}{\sqrt{(n+q)!}} \frac{\cos\vartheta_n}{\sin\vartheta_n}, \tag{9.12}$$

hence the involved combinations must be a site-independent constant

$$\frac{1}{\omega^q} \equiv \frac{\sqrt{n!}}{\sqrt{(n+q)!}} \frac{\cos\vartheta_n}{\sin\vartheta_n} = const, \tag{9.13}$$

where ω is an arbitrary real constant, which specifies the size of skyrmion.

Provided the skyrmion with (9.13) produces the absolute minimum of the energy, it must be the eigenstate of the hard-core Hamiltonian. Indeed, with the help of (8.51) and (8.52) we verify $\psi_\downarrow(\boldsymbol{r})\psi_\uparrow(\boldsymbol{r})|\mathfrak{S}^q_-\rangle = 0$ provided the relations (9.13) are held. Consequently, we have

$$H_e | \mathfrak{S}^q_- \rangle = \frac{q}{4\pi\ell^2} | \mathfrak{S}^q_- \rangle, \tag{9.14}$$

what also implies that the energy is independent of the scale parameter ω. The energy of a skyrmion (9.14) consists of direct and exchange pieces $(E_D + E_X)$. In general, the direct interaction supports large sizes in order to smear the charge over larger area, thus lowering the direct interaction energy. In contrast, the exchange interaction supports small sizes hence attempts to shrink the skyrmion. In the case of hard core interaction these rivaling effects cancel out each other and the total energy $(E_D + E_X)$ becomes scale-insensitive. Such cancelation is no longer the case for other interaction (say for Coulomb) types and the optimal size of an excitation is formed.

Remark, that the factorizable skyrmion introduced in section (8.9) by (8.108) represents the particular case of (9.13) with $q = 1$ and $\omega = \kappa$.

In previous sections we derived the expression for NC CP^1 field related with q-skyrmion state. We now study that expression by taking into account the relation of factorizability (9.13).

We start by the corresponding Weyl operators given by (8.69)

$$O[f^-_\uparrow] = \sum_{n=0}^{\infty} \cos\vartheta_n |n+q\rangle\langle n|, \tag{9.15a}$$

$$O[f^-_\downarrow] = \sum_{n=0}^{\infty} \sin\vartheta_n |n\rangle\langle n|. \tag{9.15b}$$

As a matter of (3.16) we have

$$(b^\dagger)^q|n\rangle = \frac{\sqrt{(n+q)!}}{\sqrt{n!}}|n+q\rangle, \tag{9.16}$$

what leads to

$$(b^\dagger)^q O[f_\downarrow^-] = \sum_{n=0}^{\infty} \sin\vartheta_n \frac{\sqrt{(n+q)!}}{\sqrt{n!}}|n+q\rangle\langle n|. \tag{9.17}$$

Here we use (9.13) and comparing then with (9.15a) come to

$$(b^\dagger)^q O[f_\downarrow^-] = \omega^q O[f_\uparrow^-]. \tag{9.18}$$

Remind here that $b^\dagger = O[z]$ where $\sqrt{2}\ell z = x + iy$. Also, we have $z \star z^q = z^{q+1}$. Then the above relation is equivalent to

$$z^q \star f_\downarrow^-(\boldsymbol{r}) = \omega^q f_\uparrow^-(\boldsymbol{r}), \tag{9.19}$$

and the NC CP^1 field appears as

$$f^-(\boldsymbol{r}) = \frac{1}{\omega^q}\begin{pmatrix} z^q \\ \omega^q \end{pmatrix} \star f_\downarrow^-(\boldsymbol{r}), \tag{9.20}$$

where in accord with (8.70b) $f_\downarrow^-(\boldsymbol{r})$ is the function of r only

$$f_\downarrow^-(\boldsymbol{r}) = 2e^{-r^2/\ell^2}\sum_{n=0}^{\infty}(-1)^n \sin\vartheta_n L_n(2r^2/\ell^2). \tag{9.21}$$

Remark, that we came to the structure with the spin part factorized as in (8.109). It is also remarkable that (9.20) resembles the structure of the ordinary (commutative) CP^1 field which looks as

$$f(\boldsymbol{r}) = \frac{1}{\sqrt{|z|^{2q}+\omega^{2q}}}\begin{pmatrix} z^q \\ \omega^q \end{pmatrix}. \tag{9.22}$$

Finally, we calculate the physical densities $\rho_{\rm cl}(\boldsymbol{r})$ and $I_a^{\rm cl}(\boldsymbol{r})$ for the factorizable multi-skyrmion state. From (9.13) we have

$$\sin^2\vartheta_n = \frac{\omega^{2q}}{\omega^{2q}+[(n+1)\cdots(n+q)]}, \tag{9.23a}$$

$$\cos^2\vartheta_n = \frac{(n+1)\cdots(n+q)}{\omega^{2q}+[(n+1)\cdots(n+q)]}. \tag{9.23b}$$

Substituting these into (8.56) we obtain the physical densities $\rho_{\rm cl}(\boldsymbol{r})$ and $I_a^{\rm cl}(\boldsymbol{r})$ describing q-skyrmion states to be

$$\rho_{\rm cl}(\boldsymbol{r}) = \frac{1}{2\pi\ell^2}\Big[(\tfrac{1}{2}\ell^{-2}r^2)^q + \omega^{2q}\Big]F_q(\tfrac{1}{2}\ell^{-2}r^2), \tag{9.24a}$$

$$I_x^{\rm cl}(\boldsymbol{r}) = \frac{+\cos(q\phi)}{2\pi\ell^2}\left[\frac{\omega r}{\sqrt{2}\ell}\right]^q e^{-\frac{1}{2}\ell^{-2}r^2}F_q(\tfrac{1}{2}\ell^{-2}r^2), \tag{9.24b}$$

$$I_y^{\rm cl}(\boldsymbol{r}) = \frac{-\sin(q\phi)}{2\pi\ell^2}\left[\frac{\omega r}{\sqrt{2}\ell}\right]^q e^{-\frac{1}{2}\ell^{-2}r^2}F_q(\tfrac{1}{2}\ell^{-2}r^2), \tag{9.24c}$$

$$I_z^{\rm cl}(\boldsymbol{r}) = \frac{1}{4\pi\ell^2}\Big[(\tfrac{1}{2}\ell^{-2}r^2)^q - \omega^{2q}\Big]F_q(\tfrac{1}{2}\ell^{-2}r^2), \tag{9.24d}$$

where $F_q(x)$ is given by

$$F_q(x) = e^{-x}\sum_{n=0}^{\infty}\frac{1}{[(n+1)\cdots(n+q)]+\omega^{2q}}\frac{x^n}{n!}, \tag{9.25}$$

and ϕ is the azimuthal angle.

Expressions (9.24) carry the structure

$$I_a^{\rm cl}(\boldsymbol{r}) = \rho_{\rm cl}(\boldsymbol{r})\mathcal{I}_a(\boldsymbol{r}), \tag{9.26}$$

where the field $\mathcal{I}_a(\boldsymbol{r})$ is given by

$$\mathcal{I}_x(\boldsymbol{r}) = \frac{+(\sqrt{2}\ell\omega r)^q}{r^{2q}+(2\ell^2\omega^2)^q}\cos(q\phi), \tag{9.27a}$$

$$\mathcal{I}_y(\boldsymbol{r}) = \frac{-(\sqrt{2}\ell\omega r)^q}{r^{2q}+(2\ell^2\omega^2)^q}\sin(q\phi), \tag{9.27b}$$

$$\mathcal{I}_z(\boldsymbol{r}) = \frac{1}{2}\,\frac{r^{2q}-(2\ell^2\omega^2)^q}{r^{2q}+(2\ell^2\omega^2)^q}. \tag{9.27c}$$

Being normalized as

$$\mathcal{I}_a(\boldsymbol{r})\mathcal{I}_a(\boldsymbol{r}) = \frac{1}{4}, \tag{9.28}$$

it represents the ordinary (commutative) $O(3)$ skyrmion: compare to (6.9).

Summarizing, the factorizable skyrmions represent the particular case where in addition to the general NC properties (5.33) satisfied by core fields, we also have the commutative normalization of physical spin fields. The later fact is the peculiarity of factorizable skyrmions and *does not reflect any general property*. The factorized structures (8.109) and (9.20) are also the property of the given particular case and does not occur in general case.

Though, the factorizable skyrmions exhibit interesting structures, it becomes inconsistent once the Zeeman interaction is turned on. The later reads as $H_Z = -\Delta_Z \int I_z(\boldsymbol{r})d\boldsymbol{r}$ where Δ_Z is the Zeeman gap energy. Zeeman energy of the excitation is calculated as

$$\Delta E_Z = \langle \mathfrak{S}_-|H_Z|\mathfrak{S}_-\rangle - \langle \mathrm{g}|H_Z|\mathrm{g}\rangle = -\Delta_Z \int \left[I_z^{\rm cl}(\boldsymbol{r}) - \frac{1}{4\pi\ell^2}\right] d\boldsymbol{r}, \tag{9.29}$$

where $(4\pi\ell^2)^{-1}$ is the ground state value of $S_z^{\rm cl}(\boldsymbol{r})$.

For a single-skyrmion state ($q=1$) the function (9.25) can be expressed as

$$F(z) = \frac{1}{\omega^2+1}\, e^{-z} M\big(\omega^2+1;\omega^2+2;z\big), \tag{9.30}$$

where $M(a;b;z)$ is the Kummer function. Using its properties we find the asymptotic behaviour ($z\to\infty$) for $\omega\neq 0$ to be

$$F(z) \;\to\; \frac{1}{z} - \frac{\omega^2}{z^2} + \frac{\omega^2(\omega^2-1)}{z^3} + \cdots. \tag{9.31}$$

Consequently, for $\omega\neq 0$ we obtain

$$I_z^{\rm cl}(\boldsymbol{r}) \;\to\; \frac{1}{4\pi\ell^2} - \frac{\omega^2}{\pi r^2} + \mathcal{O}(r^{-4}), \tag{9.32}$$

as $r\to\infty$. Using this relation in (9.29) we find the Zeeman energy diverges logarithmically. Zeeman energy is finite only for $\omega = 0$, *i.e.* when skyrmion turns into a hole configuration $I_z^{\rm cl}(\boldsymbol{r}) = (4\pi\ell^2)^{-1}(1-e^{-r^2/2\ell^2})$.

Summarizing, once the Zeeman interaction is turned on, the factorizable skyrmion is no longer valid in the hard-core model. There surely exists a different kind of skyrmion with a finite Zeeman energy, but this would be not a factorizable one. Even for a non-factorized skyrmion, we can show that the hole state has the lowest energy: factorizable skyrmion is the lowest energy eigenstate of the hardcore Hamiltonian. Therefore, any other state $|\mathfrak{S}\rangle$ with the same electron number $\langle\mathfrak{S}|\mathcal{N}|\mathfrak{S}\rangle - \mathcal{N}_{\rm g} = -q$ has higher energy $\langle\mathfrak{S}|H_e|\mathfrak{S}\rangle \geqslant \frac{q}{4\pi\ell^2}$. Further, its Zeeman energy is larger than that of a hole, $\langle\mathfrak{S}|H_Z|\mathfrak{S}\rangle \geqslant \frac{1}{2}\Delta_Z$. Hence

$$\langle\mathfrak{S}|(H_e+H_Z)|\mathfrak{S}\rangle \geqslant \frac{q}{4\pi\ell^2} + \frac{1}{2}\Delta_Z, \tag{9.33}$$

where the equality holds for the hole state. Consequently, in the presence of the Zeeman interaction there are no skyrmions in the hard-core interaction, so the only excitations are the hole configuration.

9.2 Coulomb Interaction

We briefly comment on the realistic system governed by the Coulomb potential

$$V(\boldsymbol{r}_1-\boldsymbol{r}_2)=\frac{\alpha}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}. \tag{9.34}$$

As pointed out in previous section the direct energy supports large sized excitation, while the exchange interaction tries to shrink it as much as possible. In the case of hard-core potential these effects canceled out each other and the total energy was scale-independent. For the Coulomb interaction such cancelation is no longer the case and the total energy is scale-dependent.

In this case it is practically impossible to construct a skyrmion as an eigenstate of the interaction Hamiltonian, so one has to estimate the excitation energy of a skyrmion by minimizing the total energy (expectation value of H_e) with respect to the scale parameter. Such an attempt has been carried out by Fertig *et. al.* (*PRB* **50**, 1994) in a model without Zeeman interaction.

As argued in previous section, inclusion of the Zeeman interaction led to logarithmic divergences. The matter is that skyrmions set by (9.13) exhibits polynomial suppression at spatial infinity what is not sufficient to produce finite Zeeman energy. As a way out of this problem one can modify the parameters ϑ_n so to endow the classical fields with exponential suppression at spatial infinity. Such configurations possess finite Zeeman energy and may be considered as a possible candidate for realistic quantum Hall skyrmion. Surely, this won't be the factorizable skyrmion. In this scope one can put for a single-skyrmion state

$$\sin\vartheta_n=\frac{\omega t^{n+1}}{\sqrt{n+\omega^2+1}}, \tag{9.35}$$

where the parameter t implements the interpolation between the hole ($t=0$) and the factorizable skyrmion ($t=1$). It is evident that the given ansatz satisfies the condition $\cos_{-1}=0$ so the results obtained in previous sections can be directly applied.

The z-component of spin field appears now as

$$4\pi\ell^2 I_z^{\mathrm{cl}}(\boldsymbol{r})=1-e^{-(1-t^2)\frac{r^2}{2\ell^2}}M\left[1;\omega^2+1;-\frac{t^2r^2}{2\ell^2}\right]-\frac{t^2\omega^2}{\omega^2+1}\,e^{-(1-t^2)\frac{r^2}{2\ell^2}}M\left[1;\omega^2+2;-\frac{t^2r^2}{2\ell^2}\right], \tag{9.36}$$

where $M(a;b;z)$ is the Kummer function.

From the asymptotic properties of Kummer functions we find that $I_z^{\mathrm{cl}}(\boldsymbol{r})$ approaches the ground state value exponentially

$$I_z^{\mathrm{cl}}(\boldsymbol{r})\;\to\;\frac{1}{4\pi\ell^2}-\frac{\omega^2}{2\pi r^2}\left[1+\frac{1}{t^2}\right]e^{-(1-t^2)\frac{r^2}{2\ell^2}}, \tag{9.37}$$

and therefore the corresponding Zeeman energy is now convergent. As expected, exponential suppression turns into the polynomial one when $t\to 1$.

Hence the energy of a skyrmion comprises two (ω and t) parameters and minimization must be carried out with respect to both of them. This cannot be done analytically and one has to employ numeric calculations. Omitting all the details we bring only the final results.

Minimization is carried out for different values of Δ_Z. In such a way we obtain the excitation energy as a function of the Zeeman coupling, what is subsequently compared to the experimental data by Schmeller *et al.* (*PRL* **75**, 1995).

Comparing theoretical results with experimental data, two points are usually taken into account. First, excitations occur in the presence of charged impurities, which reduce the excitation energy. We include an offset parameter Γ_{offset} to treat this effect phenomenologically. Second, we have so far assumed an ideal two-dimensional space for electrons. This is not the case and electrons are confined within a quantum well of a finite width. This modifies the Coulomb energy which becomes smaller than what we have assumed. It is quite difficult to make rigorous analysis of the Coulomb energy in an actual quantum well. This effect is usually taken into account by including the reduction factor for the Coulomb energy.

However, theoretical data come in good agreement with the experimental ones even without the Coulomb energy reduction. One only needs to involve the offset parameter

$$E_{\mathrm{gap}}\;\to\;E_{\mathrm{gap}}-\Gamma_{\mathrm{offset}}\,. \tag{9.38}$$

Experimental data were obtained in three samples (figure 9.1): a single heterointerface (✦) and two $GaSa$ single quantum wells (✧,✶ and ✷) with widths of 20nm and 14nm. The sample ✦ has much wider thickness of the two-dimensional sheet. In figure 9.1 the sample mobilities (planar density of free electrons) are 3.4, 0.52, 0.18, and 0.16

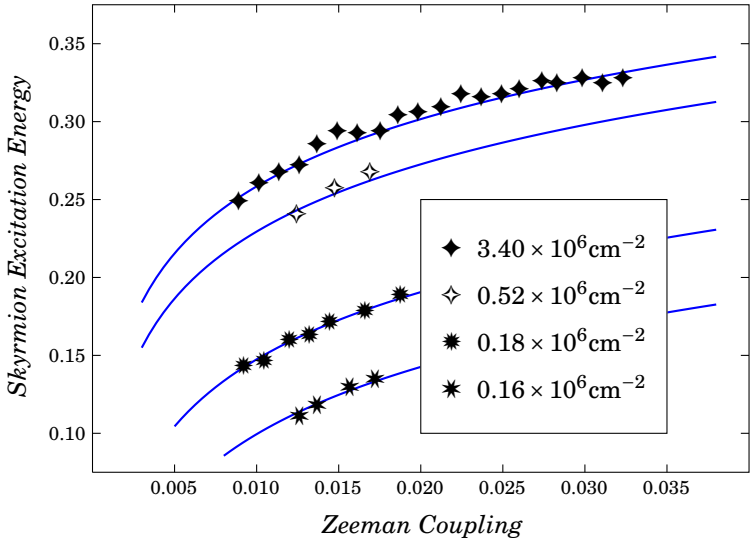


Fig. 9.1: Skyrmion excitation energies and Zeeman coupling are measured in units of the Coulomb scale $E_C = \alpha/\ell$. Applying the parallel magnetic field, the Coulomb energy is not affected since it depends only on the orthogonal component $B_\perp$, while the electron spin feels the total magnetic field $(B_\perp^2 + B_\parallel^2)^{\frac{1}{2}}$ what results in the increase of Zeeman interaction. In this way experimentalists manipulate with Zeeman coupling what enables to study spin effects in quantum Hall systems.

($\times 10^6 \mathrm{cm}^{-2}$), respectively. Theoretical results successfully fit the experimental data by the appropriate choices of the offset parameter Γ_{offset} in figure 9.1. Different values of Γ_{offset} are used for different samples.